Asset Pricing Tests with Long-run Risks in Consumption Growth

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We present a novel methodology for estimating/testing the Bansal and Yaron (2004) and related long-run risks (LRR) models based on the observation that the latent state variables are known functions of observables. The large standard error of the estimated elasticity of intertemporal substitution explains the controversy on its magnitude. The model requires higher persistence of consumption and dividend growth to explain the cross-section of returns than that observed in the data. The model matches the unconditional moments of consumption and dividend growth, but implies a higher risk-free rate and lower volatility of the price/dividend ratio, risk-free rate, and market return than those observed in the data. Contrary to the model implications, the conditional variance of the LRR variable fails to capture the large time variation in the equity premium. (*JEL* G12, E44)

Introduction

A burgeoning literature in finance addresses investors' attitudes toward the timing of resolution of uncertainty of future consumption and cash flows through the class of preferences introduced by Epstein and Zin (1989), Kreps and Porteus (1978), and Weil (1989). Models initiated by Bansal, Dittmar, and Lundblad (2005), Bansal and Yaron (2004), and Hansen, Heaton, and Li (2008) have rich implications on prices and show promise in explaining the time-series and cross-sectional properties of returns of financial assets. These models pay particular attention to the low-frequency properties of the time series of dividends and aggregate consumption—hence their characterization as long-run

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We thank Ravi Bansal, Alan Bester, John Campbell, Raj Chakrabarti, John Cochrane, Eugene Fama, Wayne Ferson (the editor), Lars Hansen, Christian Julliard, Dana Kiku, Ralph Koijen, Oliver Linton, Sydney Ludvigson, Alex Michaelides, Lubos Pastor, Annette Vissing-Jorgensen, and Amir Yaron (the editor) for helpful comments. Constantinides acknowledges financial support from the Center for Research in Security Prices of the University of Chicago Booth School of Business. Ghosh acknowledges financial support from the London School of Economics. Send correspondence to George M. Constantinides, Booth School of Business, University of Chicago, and NBER; 5807 South Woodlawn Avenue, Chicago IL 60637; telephone: (773)702-7258; fax: (773)753-8045. E-mail: gmc@chicagobooth.edu; http://faculty.chicagobooth.edu/george.constantinides/.

doi:10.1093/rapstu/rar004 Advance Access publication October 17, 2011

risks (LRR) models.[1] The main difficulty in assessing their empirical plausibility is their reliance on latent state variables.

We propose an empirical methodology for estimating and testing asset pricing models of the cross-section of equity returns when the state variables are latent. We apply this methodology to revisit the log-linearized LRR model introduced by Bansal and Yaron (2004) (hereafter B-Y) and provide novel insights into this class of models. The latent state variables, the conditional mean of the aggregate consumption growth rate (the LRR variable) and the conditional variance of its innovation, are hard to measure in the data. We bypass the need to filter the latent state variables, a procedure that potentially introduces estimation error and decreases the power of the tests. We argue that the two latent state variables are, in fact, observable because both the aggregate log price/dividend ratio and log risk-free rate are affine functions of only these two state variables with coefficients that are known functions of the preference parameters and of the parameters of the time-series processes. This observation allows us to invert the system and express the two state variables as known affine functions of the observable aggregate log price/dividend ratio and log risk-free rate. Whereas this methodology is common in the context of affine term structure models (for example, Dai and Singleton 2000 and Duffee 2002), this is the first application in the equities literature.

We estimate the model using the Generalized Method of Moments (GMM) approach on the joint system of the Euler equations of consumption and the restrictions imposed on the model parameters by the unconditional moments of the aggregate dividend and consumption growth over 1931–2009. We are able to write down the Euler equations without reference to the latent state variables because we express the log pricing kernel as an affine function of the aggregate log price/dividend ratio, the log risk-free rate, and their lags, in addition to consumption growth. The estimated parameter values and, most importantly, their standard errors provide insights beyond those obtainable via calibration.

The most notable finding is that the standard error of the estimated intertemporal elasticity of substitution in consumption is large. One cannot reject either the hypothesis that it is lower than one or the hypothesis that it exceeds one. Furthermore, one cannot reject the hypothesis that it is either lower or higher than the inverse of the risk-aversion coefficient. Therefore, these results offer an insight as to why the magnitude of the elasticity is controversial in the literature. The results suggest that one should explore LRR models with a wide range of values for the elasticity.

1 For further references, see Alvarez and Jermann (2005), Bansal, Dittmar, and Kiku (2009), Bansal, Gallant, and Tauchen (2007), Bansal, Kiku, and Yaron (2010), Bansal and Shaliastovich (2010), Beeler and Campbell (2011), Bekaert, Engstrom, and Xing (2009), Chen, Favilukis, and Ludvigson (2011), Colacito and Croce (2011), Croce, Lettau, and Ludvigson (2010), Drechsler and Yaron (2011), Ferson, Nallareddy, and Xie (2011), Ghosh and Constantinides (2011), Hansen and Scheinkman (2009), Jagannathan and Marakani (2010), Lettau and Ludvigson (2009), Lustig, Van Nieuwerburgh, and Verdelhan (2008), Malloy, Moskowitz, and Vissing-Jorgensen (2009), Parker and Julliard (2005), and Piazzesi and Schneider (2006).

Another finding is that the model requires higher persistence of consumption and dividend growth to explain the cross-section of returns over the period 1931–2009 than the persistence estimated from the time series of consumption and dividend growth alone. This suggests that one should explore channels through which a lower level of persistence can address the cross-section of equity returns.

In simultaneously testing the Euler equations of consumption and the restrictions imposed on the model parameters by the unconditional moments of the aggregate dividend and consumption growth, we find that the model matches the unconditional moments of the aggregate consumption and dividend growth rates. Therefore, the model is on the right track. However, it implies a higher value for the risk-free rate than that observed in the data (2.8%–4.5% versus 0.6%), lower volatility of the risk-free rate (0.9%–1.7% versus 3.0%), and lower volatility of the marketwide price/dividend ratio (0.11–0.19 versus 0.45). Moreover, it implies economically large annual pricing errors for the "Small" capitalization and the "Value" portfolios. An implication of these findings is that one should explore ways to enhance the model by refining the definition of the state variables and possibly introducing additional ones.

Finally, we address the model's implications regarding predictability. The model implies that the conditional expectation of the equity premium is an affine function of the conditional variance of the LRR variable, yet we find that the conditional variance does not predict the equity premium. We also find that the LRR variable predicts the equity premium, despite the implications of the model to the contrary, suggesting that the model may be enhanced either by making the conditional expectation of the equity premium dependent on state variables other than the conditional variance of the LRR variable or by an alternative specification of the dynamics of the conditional variance process. We verify the model's implication that the LRR variable predicts consumption and dividend growth. However, the fact that the conditional variance also contributes in predicting consumption and dividend growth, even though the model does not imply such predictability, suggests that the model may be enhanced in ways that make the conditional expectation of consumption and dividend growth dependent on state variables in addition to the LRR variable. Whereas these predictability results may be partly due to estimation error in the model parameters, we argue in Section 4 that this is unlikely to be the full explanation.

In our second application of the methodology, we revisit the co-integrated extension of the B-Y model by Bansal, Gallant, and Tauchen (2007), which introduces the aggregate consumption-to-dividend ratio as a third state variable. The two latent state variables are observable because both the aggregate log price/dividend ratio and log risk-free rate are affine functions of only the two latent state variables and the observable consumption-to-dividend ratio with coefficients that are known functions of the preference parameters and of the parameters of the time-series processes. This observation allows us to invert the system and express the two latent state variables as known affine functions of the observable aggregate log price/dividend ratio, log risk-free rate, and consumption-to-dividend ratio. The conclusions are broadly similar to those for the B-Y model.

We address the possibility that the decision interval may be monthly instead of annual by comparing our estimation and testing results at the annual frequency to those obtained using the B-Y monthly calibration. The results, available from the authors upon request, are very similar, suggesting that our findings are unlikely to be driven by the choice of the decision frequency.

The article is organized as follows. In Section 1, we describe the estimation and testing methodology of the B-Y model. We discuss the data in Section 2. In Section 3, we estimate the model, discuss the parameter estimates, present the empirical evidence on the cross-section of returns, and explore the robustness of the results. In Section 4, we present the results of the model-implied in-sample forecasting regressions for the equity premium and the aggregate consumption and dividend growth rates. In Section 5, we estimate and test the co-integrated extension of the model. Section 6 concludes. The appendix contains derivations and details of the testing methodology.

1. The Model and Its Testable Implications

We describe the LRR model of Bansal and Yaron (2004) and derive its testable implications for the equity premium and the cross-section of returns. Then we derive its testable implications for the predictability of the equity premium, dividend growth, and consumption growth.

1.1 Model

The Bansal and Yaron (2004) LRR model introduces the novel state variable, x_t, and the variance of its innovation, σ_t^2, that jointly drive the conditional mean of the aggregate consumption and dividend growth rates:

$$x_{t+1} = \rho_x x_t + \psi_x \sigma_t \varepsilon_{x,t+1}, \tag{1}$$

$$\sigma_{t+1}^2 = (1-\upsilon)\sigma^2 + \upsilon\sigma_t^2 + \sigma_w \varepsilon_{\sigma,t+1}, \tag{2}$$

$$\Delta c_{t+1} = \mu_c + x_t + \sigma_t \varepsilon_{c,t+1}, \tag{3}$$

$$\Delta d_{t+1} = \mu_d + \phi x_t + \varphi \sigma_t \varepsilon_{d,t+1}, \tag{4}$$

where c_{t+1} is the logarithm of the aggregate consumption level and d_{t+1} is the logarithm of the aggregate stock market dividends. The shocks $\varepsilon_{x,t+1}$, $\varepsilon_{\sigma,t+1}$, $\varepsilon_{c,t+1}$, and $\varepsilon_{d,t+1}$ are assumed to be $i.i.d.\ N(0,1)$ and mutually independent. The time-series specification in Equations (1)–(4) introduces nine parameters: μ_c, μ_d, ϕ, φ, ρ_x, ψ_x, σ, υ, and σ_w. In Appendix A.1, we derive various unconditional moments of consumption and dividend growth rates as functions of the time-series parameters.

The model further assumes that the representative consumer has the version of Kreps and Porteus (1978) preferences adopted by Epstein and Zin (1989), and Weil (1989). These preferences allow for separation between the coefficient of risk aversion and the elasticity of intertemporal substitution. The utility function is defined recursively as

$$V_t = \left[(1-\delta)C_t^{\frac{1-\gamma}{\theta}} + \delta\left(E_t\left[V_{t+1}^{1-\gamma}\right]\right)^{\frac{1}{\theta}}\right]^{\frac{\theta}{1-\gamma}}, \tag{5}$$

where δ denotes the subjective discount factor, $\gamma > 0$ is the coefficient of risk aversion, $\psi > 0$ is the elasticity of intertemporal substitution, and $\theta = \frac{1-\gamma}{1-\frac{1}{\psi}}$. Note that the sign of θ depends on the relative magnitudes of γ and ψ. The standard time-separable power utility model is obtained as a special case when $\theta = 1$, i.e., $\gamma = \frac{1}{\psi}$.

For this specification of preferences, Epstein and Zin (1989) and Weil (1989) show that, for any asset j, the first-order conditions of the consumer's utility maximization yield the Euler equation,

$$E_t\left[\exp(m_{t+1} + r_{j,t+1})\right] = 1, \tag{6}$$

where

$$m_{t+1} = \theta \log \delta - \frac{\theta}{\psi}\Delta c_{t+1} + (\theta - 1) r_{c,t+1} \tag{7}$$

is the natural logarithm of the intertemporal marginal rate of substitution; $E_t[.]$ denotes expectation conditional on time t information; $r_{j,t+1}$ is the continuously compounded return on asset j; and $r_{c,t+1}$ is the unobservable continuously compounded return on an asset that delivers aggregate consumption as its dividend each period.

We rely on log-linear approximations for the log return on the consumption claim, $r_{c,t+1}$, and on the market portfolio (the return on the aggregate dividend claim), $r_{m,t+1}$, as in Campbell and Shiller (1988):

$$r_{c,t+1} = \kappa_0 + \kappa_1 z_{t+1} - z_t + \Delta c_{t+1}, \tag{8}$$

$$r_{m,t+1} = \kappa_{0,m} + \kappa_{1,m} z_{m,t+1} - z_{m,t} + \Delta d_{t+1}, \tag{9}$$

where z_t is the log price/consumption ratio and $z_{m,t}$ the log price/dividend ratio. In Equation (8), $\kappa_1 = \frac{e^{\overline{z}}}{1+e^{\overline{z}}}$ and $\kappa_0 = \log(1+e^{\overline{z}}) - \kappa_1 \overline{z}$ are log-linearization constants, where $\overline{z}$ denotes the long-run mean of the log price/consumption ratio. Similarly, in Equation (9), $\kappa_{1,m} = \frac{e^{\overline{z}_m}}{1+e^{\overline{z}_m}}$ and $\kappa_{0,m} = \log(1+e^{\overline{z}_m}) - \kappa_{1,m}\overline{z}_m$, where $\overline{z}_m$ denotes the long-run mean of the log price/dividend ratio.

B-Y show that z_t and $z_{m,t}$ are affine functions of the state variables x_t and σ_t^2,

$$z_t = A_0 + A_1 x_t + A_2 \sigma_t^2, \tag{10}$$

$$z_{m,t} = A_{0,m} + A_{1,m} x_t + A_{2,m} \sigma_t^2. \tag{11}$$

The coefficients A_0, A_1, A_2, $A_{0,m}$, $A_{1,m}$, and $A_{2,m}$ depend on the parameters of the utility function, those of the stochastic processes for consumption and dividend growth rates, and the linearization parameters, κ_0, κ_1, $\kappa_{0,m}$, and $\kappa_{1,m}$ (see Appendix A.2.1 for expressions for these coefficients and for the procedure that ensures that the linearization parameters κ_0, κ_1, $\kappa_{0,m}$, and $\kappa_{1,m}$ are consistent with Equations (10) and (11)).

For this model specification, the log risk-free rate from period t to $t+1$ may also be expressed as an affine function of the state variables (see Appendix A.2.2 for expressions for $A_{0,f}$, $A_{1,f}$, and $A_{2,f}$),

$$\begin{aligned} r_{f,t} &= -\log E_t\left[\exp(m_{t+1})\right], \\ &= A_{0,f} + A_{1,f}x_t + A_{2,f}\sigma_t^2. \end{aligned} \tag{12}$$

Equations (11) and (12) express the observable variables, $z_{m,t}$ and $r_{f,t}$, as affine functions of the latent state variables, x_t and σ_t^2. These equations may be *inverted* to express the latent state variables, x_t and σ_t^2, as affine functions of the observables, $z_{m,t}$ and $r_{f,t}$ (see Appendix A.2.3 for details and expressions for α_0, α_1, α_2, β_0, β_1, and β_2),

$$x_t = \alpha_0 + \alpha_1 r_{f,t} + \alpha_2 z_{m,t}, \tag{13}$$

$$\sigma_t^2 = \beta_0 + \beta_1 r_{f,t} + \beta_2 z_{m,t}. \tag{14}$$

1.2 Testable implications for the equity premium and the cross-section of returns

Substituting the log-affine approximation for $r_{c,t+1}$ in Equation (8) into the expression for the pricing kernel (Equation (7)), and noting that z_t is given by Equation (10), we have

$$\begin{aligned} m_{t+1} = {} & (\theta \log\delta + (\theta-1)\left[\kappa_0 + (\kappa_1 - 1)A_0\right]) + \left(-\frac{\theta}{\psi} + (\theta-1)\right)\Delta c_{t+1} \\ & + (\theta-1)\kappa_1 A_1 x_{t+1} + (\theta-1)\kappa_1 A_2 \sigma_{t+1}^2 - (\theta-1)A_1 x_t - (\theta-1)A_2\sigma_t^2. \end{aligned} \tag{15}$$

Equation (15) for the pricing kernel involves the unobservable (from the point of view of the econometrician) state variables x_t and σ_t^2, and, hence, is not directly testable on a cross-section of asset returns. Substituting the expressions for x_t and σ_t^2 from Equations (13) and (14) into the pricing kernel in Equation (15), we have

$$m_{t+1} = c_1 + c_2\Delta c_{t+1} + c_3\left(r_{f,t+1} - \frac{1}{\kappa_1}r_{f,t}\right) + c_4\left(z_{m,t+1} - \frac{1}{\kappa_1}z_{m,t}\right). \tag{16}$$

The parameters $c = (c_1, c_2, c_3, c_4)'$ are functions of the parameters of the time-series processes and the preference parameters (see Appendix A.2.4 for details). The above expression for the pricing kernel is entirely in terms of observables. We substitute this expression into the set of Euler equations (6) to obtain a set of moment restrictions that are expressed entirely in terms of observables.

We first examine the empirical plausibility of the model when the set of assets consists of the market portfolio and the risk-free rate, thereby focusing on the equity premium and risk-free rate puzzles. To the set of their Euler equations we add restrictions on the unconditional moments of consumption and dividend growth implied by the time-series specification of the model. We estimate the parameters with GMM and test the specification of the model with the overidentifying restrictions. We then examine the ability of the model to explain the cross-section of returns. The set of assets consists of the "Value," "Growth," "Small" capitalization, and "Large" capitalization portfolios, in addition to the market portfolio and the risk-free rate. To the set of their Euler equations we add moment restrictions implied by the time-series specification of the model and test with GMM.

1.3 Testable implications for predicting returns and growth rates

Equations (9), (11), (4), and (12) imply that the equilibrium expected market return is an affine function of the state variables x_t and σ_t^2:

$$E_t[r_{m,t+1}] = B_0 + B_1 x_t + B_2 \sigma_t^2; \tag{17}$$

and the expected equity premium is an affine function of the state variable σ_t^2 alone:

$$E_t[r_{m,t+1} - r_{f,t}] = E_0 + E_1 \sigma_t^2. \tag{18}$$

The coefficients are known functions of the underlying time-series and preference parameters.

The model also implies that the conditional variance of the market return is an affine function of the state variable σ_t^2:

$$Var_t\left(r_{m,t+1}\right) = \left(\kappa_{1,m} A_{2,m} \sigma_w\right)^2 + \left[\left(\kappa_{1,m} A_{1,m} \psi_x\right)^2 + \varphi_d^2\right] \sigma_t^2. \tag{19}$$

Finally, the time-series specification of the model implies that the expected consumption growth rate is given by

$$E_t[\Delta c_{t+1}] = \mu_c + x_t, \tag{20}$$

and the expected dividend growth rate is given by

$$E_t[\Delta d_{t+1}] = \mu_d + \phi x_t, \tag{21}$$

both affine functions of the state variable x_t.

Since the state variables x_t and σ_t^2 are affine functions of the observables $z_{m,t}$ and $r_{f,t}$, we use the point estimates of the time-series and preference parameters and the time series of $z_{m,t}$ and $r_{f,t}$ to extract the time series of the state variables. In Section 4, we test the predictive implications of the model through in-sample linear forecasting regressions of the realized equity premium on the state variable σ_t^2 and of the aggregate consumption and dividend growth rates on the LRR variable x_t.

2. Data

We use monthly data on prices and dividends and annual data on consumption from January 1929 through December 2009. The proxy for the market is the Center for Research in Security Prices (CRSP) value-weighted index of all stocks on the NYSE, AMEX, and NASDAQ. The construction of the size and book-to-market portfolios is as in Fama and French (1993). In particular, for the size sort, all NYSE, AMEX, and NASDAQ stocks are allocated across ten portfolios in June of each year according to their market capitalization at the end of June. NYSE breakpoints are used in the sort. Value-weighted monthly returns on these size-sorted portfolios are computed from July of the year to June of the next year. “Small” and “Large” denote the bottom and top market capitalization deciles, respectively. For the book-to-market equity sort, all NYSE, AMEX, and NASDAQ stocks are allocated across ten portfolios in June of each year according to their book equity (BE) to market equity (ME) ratio at the end of the previous year. NYSE breakpoints are used in the sort. Value-weighted monthly returns on these BE/ME-sorted portfolios are computed from July of the year to June of the next year. “Growth” and “Value” denote the bottom and top BE/ME deciles, respectively.

The monthly portfolio return is the sum of the portfolio price and dividends at the end of the month, divided by the portfolio price at the beginning of the month. The annual portfolio return is the sum of the portfolio price at the end of the year and uncompounded dividends over the year, divided by the portfolio price at the beginning of the year. The real annual portfolio return is the above annual portfolio return deflated by the realized growth in the Consumer Price Index.

The proxy for the real annual risk-free rate is obtained as in Beeler and Campbell (2011). Specifically, the quarterly nominal yield on three-month Treasury bills is deflated using the realized growth in the Consumer Price Index to obtain the ex post real three-month Treasury-bill rate. The ex ante quarterly risk-free rate is then obtained as the fitted value from the regression of the ex post three-month Treasury-bill rate on the three-month nominal yield and the realized growth in the Consumer Price Index over the previous year. Finally, the ex ante quarterly risk-free rate at the beginning of the year is annualized to obtain the ex ante annual risk-free rate.

The annual price/dividend ratio of the market is the market price at the end of the year divided by the sum of dividends over the previous twelve months. The dividend growth rate is the sum of dividends over the year divided by the sum of dividends over the previous year, and it is deflated using the realized growth in the Consumer Price Index.

Table 1
Summary statistics, 1931–2009

	Mean	*Std. Dev.*	*AC*(1)
r_m	0.062 (0.019)	0.198 (0.017)	−0.068 (0.087)
r_f	0.006 (0.005)	0.030 (0.005)	0.672 (0.216)
r_s	0.103 (0.038)	0.333 (0.031)	0.086 (0.097)
r_b	0.056 (0.019)	0.187 (0.015)	−0.002 (0.090)
r_g	0.050 (0.022)	0.212 (0.018)	−0.027 (0.106)
r_v	0.095 (0.028)	0.299 (0.029)	−0.124 (0.085)
$log(P/D)$	3.38 (0.080)	0.45 (0.051)	0.877 (0.231)
Δd	0.010 (0.013)	0.117 (0.020)	0.163 (0.136)
Δc	0.020 (0.003)	0.021 (0.004)	0.449 (0.242)

The table reports the sample mean, volatility, and first-order autocorrelation (Newey-West asymptotic standard errors with two lags in parentheses) of the annual log market return, risk-free rate, the "Small," "Large," "Growth," and "Value" portfolio returns, the marketwide log price/dividend ratio, and the log dividend and consumption growth rates.

Consumption data are obtained from the Bureau of Economic Analysis. The real annual consumption growth rate is the real per-capita personal consumption expenditure on nondurable goods and services over the year divided by the per-capita personal consumption expenditure on nondurable goods and services over the previous year.

Table 1 provides descriptive statistics for the continuously compounded returns on the assets, the marketwide price/dividend ratio, and the aggregate consumption and dividend growth rates for the annual sample over the period 1931–2009. The table illustrates the well-documented equity premium and the size and value premia. Over the sample period, the annual equity premium over the risk-free rate has mean 5.6% and the volatility of the market return is 19.8%. The annual risk-free rate has mean 0.6% and volatility 3.0%. The annual mean premium of small over large stocks is 4.7%, and of value over growth stocks is 4.5%. Value stocks are more volatile than growth stocks, and small stocks are much more volatile than large stocks.

The annual log price/dividend ratio on the market has mean 3.38 and volatility 0.45 over the sample period. The average annual log dividend growth rate on the market portfolio is 1.0% with volatility 11.7%. Finally, the annual log consumption growth has mean 2.0% and volatility 2.1% over the sample period.

3. Parameter Estimates and Model-generated Moments

3.1 Parameter estimates from the time-series processes

We estimate the parameters of the time-series processes of aggregate consumption and dividend growth over 1931–2009, without reference to the Euler equations. We estimate the nine parameters of the time-series model (1)–(4) to match the following nine sample moments: the unconditional mean, variance, and first-order autocorrelation of consumption and dividend growth rates, the correlation between consumption and dividend growth rates, and the variance of squared consumption and dividend growth rates. These estimates serve as a benchmark for comparison when we subsequently reestimate these parameters from the joint system of the time-series moment restrictions and the Euler equations.

The point estimates, along with the associated standard errors Newey and West (1987) corrected using two lags in parentheses, are displayed in the first row of Table 2. Note that the system is exactly identified and, therefore, the model-generated moments computed at the point estimates of the parameters closely match their sample analogs. The estimated parameter values and, most importantly, their standard errors provide insights beyond those obtainable via calibration. The point estimate of the persistence parameter (ρ_x) of the LRR variable is 0.44 and is significantly different from zero. This finding lends support to the major risk channel highlighted in the LRR literature—a predictable component in the aggregate consumption and dividend growth rates. The parameter ϕ, which measures the sensitivity of the expected dividend growth rate to changes in the LRR variable, is statistically significant, while the parameters φ and ψ_x, which determine the volatility of the innovations to dividend growth and the LRR variable, respectively, are very imprecisely estimated. Finally, the parameters governing the dynamics of the conditional variance process in Equation (2), namely (σ, υ, σ_w), are imprecisely estimated and none of them is significantly different from zero. This imprecision may be due to the lack of power or misspecification of the dynamics of the volatility process. In Section 4, we find support for the latter by showing that the conditional variance (σ_t^2) does not forecast the equity premium, contrary to the implications of the model.

3.2 Parameter estimates and model-generated moments from the time-series processes and the two-asset system

We reestimate the parameters of the time-series processes of aggregate consumption and dividend growth along with the preference parameters over 1931–2009 from the joint system of the nine unconditional moments implied by the time-series processes and six Euler equations of consumption on the market return and risk-free rate. This enables us to address the ability of the model to explain the equity premium and risk-free rate puzzles. We are able to write down Euler equations without reference to the latent state variables because we express the log pricing kernel as an affine function of the aggregate log price/dividend ratio, the log risk-free rate, and their lags, in addition to consumption growth. We augment the two unconditional Euler equations for the market return and the risk-free rate with four Euler equations conditional on the

Table 2
Estimation of the B-Y LRR model on the two-asset system, 1931–2009

Parameter	δ	γ	ψ	μ_c	μ_d	ϕ	φ	ρ_x	ψ_x	σ	υ	σ_w
$Est.^{time-series}$ (*Std. Err.*)	–	–	–	0.020 (0.003)	0.010 (0.012)	2.06 (0.54)	15.8 (140.8)	0.437 (0.199)	5.20 (47.8)	0.006 (0.049)	0.208 (2.667)	6.0×10^{-5} (0.001)
$Est.^{full\ model}$ (*Std. Err.*)	0.968 (0.057)	9.34 (4.82)	1.41 (2.94)	0.021 (0.002)	0.018 (0.011)	5.14 (1.79)	3.06 (4.15)	0.482 (0.261)	0.900 (0.714)	0.012 (0.003)	0.304 (1.624)	4.6×10^{-4} (5.7×10^{-4})

Moments	*Data*	*Model*
$E(\Delta c_{t+1})$	0.020 (0.003)	0.021 [.014,.028]
$\sigma(\Delta c_{t+1})$	0.021 (0.004)	0.018 [.017,.030]
$AC1(\Delta c)$	0.449 (0.242)	0.248 [−.054,.479]
$E(\Delta d_{t+1})$	0.010 (0.013)	0.018 [−.016,.052]
$\sigma(\Delta d_{t+1})$	0.117 (0.020)	0.075 [.071,.130]
$AC1(\Delta d)$	0.163 (0.136)	0.361 [.054,.564]
$AC1(\Delta c, \Delta d)$	0.637 (0.306)	0.620 [.368,.782]
$\sigma^2\left[(\Delta c_{t+1})^2\right]$	1.0×10^{-6} (4.9×10^{-7})	2.4×10^{-6} $[5.6\times10^{-7}, 6.4\times10^{-6}]$
$\sigma^2\left[(\Delta d_{t+1})^2\right]$	0.0013 (0.0007)	0.0006 [0.0001,.0015]
J-stat	9.45 (0.024)	

Moments	*Data*	*Model*
$E(r_m)$	0.062 (0.019)	0.058 [.027,.091]
$\sigma(r_m)$	0.198 (0.017)	0.107 [.096,.171]
$E(r_f)$	0.006 (0.005)	0.045 [.040,.049]
$\sigma(r_f)$	0.030 (0.005)	0.010 [.009,.016]
$E(z_m)$	3.38 (0.080)	3.20 [3.12,3.23]
$\sigma(z_m)$	0.45 (0.051)	0.112 [.098,.186]

The table reports GMM estimates (asymptotic standard errors in parentheses) of the model parameters defined in Section 1. It also reports the model-implied (95% confidence interval in square brackets obtained through 10,000 simulations) and the historical values (asymptotic standard errors in parentheses) of the mean and volatility of the risk-free rate, price/dividend ratio, and market return, and unconditional moments of the consumption and dividend growth rates. Finally, it reports the J-stat for the overidentifying restrictions along with the associated asymptotic p-value in parentheses.

lagged log price/dividend ratio of the market and the lagged log risk-free rate. Combined with the nine time-series moment restrictions, this system of fifteen restrictions and twelve parameters (nine time-series parameters plus three preference parameters) is overidentified. We estimate the parameters with GMM using the efficient weighting matrix and test the model with the overidentifying restrictions.[2] The point estimates, along with the associated standard errors in parentheses, are displayed in the second row of Table 2. We also verify the robustness of the estimation and tests by replacing the efficient weighting matrix with the identity matrix in Section 3.4.2.

The most notable finding is that the standard error of the estimated intertemporal elasticity of substitution in consumption (ψ) is large, and one can reject neither the hypothesis that it is lower than one nor that it exceeds one, thereby providing an insight as to why the magnitude of the elasticity is a controversial issue in the literature. This lack of precision should be contrasted with the plausible and relatively precise estimates of the subjective discount factor (δ) and relative risk aversion coefficient (γ). The lack of precision in estimating the elasticity suggests that one should explore LRR models with a wide range of values for this parameter.

In Table 2, we also report the historical and model-generated moments of the consumption and dividend growth rates, market return, risk-free rate, and marketwide price/dividend ratio. The “Data” column reports the moments computed from historical data along with standard errors in parentheses. The “Model” column presents the model-generated moments along with the 95% confidence intervals in square brackets. We calculate the model-generated moments from the analytical expressions for these moments at the point estimates of the parameters. We calculate their 95% confidence intervals from 10,000 simulations of eighty years each, the same size as the historical sample. The model does a good job at matching the unconditional moments of the aggregate consumption and dividend growth rates and the mean market return. However, it implies a higher value for the risk-free rate than that observed in the data (4.5% versus 0.6%), lower volatility of the risk-free rate (1.0% versus 3.0%), and lower volatility of the market return (10.7% versus 19.8%).

The model also implies a lower volatility of the marketwide price/dividend ratio (0.11 versus 0.45) as noted earlier in Beeler and Campbell (2011). The reason for this can be explained as follows. The price/dividend ratio is an affine function of the two state variables (Equation (11)). Using the point estimates of the parameters in Table 2, most of the variability of the price/dividend ratio in the model is due to variation in the LRR variable (85.9%), with the conditional variance of the LRR variable accounting for only 14.1% of the variance of the price/dividend ratio. Therefore, the volatility of the price/dividend ratio is largely determined by the persistence parameter of the LRR variable and

[2] The numerical search for a global minimum is done using the library “DEoptim” that is built in the statistical package R. An independent grid search algorithm produces very similar results.

the elasticity of intertemporal substitution, which determine the loading of the price/dividend ratio on the LRR variable. The point estimate of the persistence parameter of the LRR variable is 0.48 in Table 2, which gives rise to a volatility of 0.11 for the price/dividend ratio. For example, if we choose the persistence parameter to be 0.88, the model-implied volatility of the price/dividend ratio becomes 0.44 and closely matches the observed volatility of 0.45. However, this counterfactually implies a much higher persistence of the consumption growth rate than that observed in the data (0.69 versus 0.45) and much higher persistence of the dividend growth rate (0.80 versus 0.16).[3]

Finally, the J-stat is 9.45 and has asymptotic p-value 2.4%. Overall, these results suggest that one should explore ways to further enhance the model by refining the definition of the state variables and possibly introducing additional state variables.

3.3 Parameter estimates and model-generated moments from the time-series processes and the six-asset system

We augment the set of assets to include the “Value,” “Growth,” “Small” capitalization, and “Large” capitalization portfolios, in addition to the market portfolio and the risk-free rate. The unconditional Euler equations for these six assets along with the nine time-series moment restrictions give fifteen moment restrictions in twelve parameters. The model does a better job at pricing some unconditional moments than others. The large standard errors of the estimated preference parameters and the parameters governing the conditional variance process, σ_t^2, neither lend support nor deny the possibility that the channels of high elasticity and the particular conditional variance process in the B-Y model are pivotal in addressing the cross-section of returns. The results are reported in Table 3.

The point estimate of the persistence parameter of the LRR variable is 0.75 and is much higher than the value of 0.44 estimated from the time series of consumption and dividend growth alone. Therefore, the model requires much higher persistence of consumption and dividend growth to explain the cross-section of returns than the persistence estimated from the time series of the growth rates. The point estimate of the elasticity (1.82) and its standard error are both higher than the corresponding values in Table 2. These findings reinforce the earlier conclusion that one should explore LRR models with a wide range of values for the elasticity.

[3] We also examined the B-Y model under the interpretation that the decision frequency is monthly. Since we do not have reliable monthly data to directly test the model at the monthly frequency, we adopted the B-Y calibration at the monthly frequency and computed the model-implied annual moments via simulation. The results are available from the authors upon request. The model does a better job than the model in Table 2 in matching the volatility of dividend growth, market return, risk-free rate, and price/dividend ratio. However, the model does a worse job than the model in Table 2 in matching the volatility of consumption growth, the correlation between consumption and dividend growth, and the mean of the price/dividend ratio. Overall, the results suggest that the interpretation of the model at the monthly frequency improves the ability of the model to match certain moments of the data compared with an annual decision interval while worsening the model’s fit for certain other moments.

Table 3
Estimation of the B-Y LRR model on the six-asset system, 1931–2009

Parameter	δ	γ	ψ	μ_c	μ_d	ϕ	φ	ρ_x	ψ_x	σ	υ	σ_w
$Est.^{time-series}$	–	–	–	0.020	0.010	2.06	15.8	0.437	5.20	0.006	0.208	6.0×10^{-5}
(*Std. Err.*)				(0.003)	(0.012)	(0.54)	(140.8)	(0.199)	(47.8)	(0.049)	(2.667)	(0.001)
$Est.^{full\ model}$	0.978	7.82	1.82	0.018	0.017	3.51	5.75	0.75	0.83	0.011	0.279	0.0002
(*Std. Err.*)	(0.262)	(7.29)	(5.40)	(0.003)	(0.011)	(1.09)	(4.70)	(0.52)	(1.44)	(0.005)	(2.789)	(0.0004)

Moments	*Data*	*Model*	*Moments*	*Data*	*Model*
$E(\Delta c_{t+1})$	0.020 (0.003)	0.018 [.009,.027]	$E(r_m)$	0.062 (0.019)	0.048 [.015,.082]
$\sigma(\Delta c_{t+1})$	0.021 (0.004)	0.017 [.014,.025]	$\sigma(r_m)$	0.198 (0.017)	0.121 [.101,.170]
$AC1(\Delta c)$	0.449 (0.242)	0.463 [.112,.649]	$E(r_f)$	0.006 (0.005)	0.030 [.023,.033]
$E(\Delta d_{t+1})$	0.010 (0.013)	0.017 [−.018,.051]	$\sigma(r_f)$	0.030 (0.005)	0.009 [.006,.012]
$\sigma(\Delta d_{t+1})$	0.117 (0.020)	0.077 [.067,.113]	$E(z_m)$	3.38 (0.080)	3.44 [3.34,3.53]
$AC1(\Delta d)$	0.163 (0.136)	0.281 [−.049,.497]	$\sigma(z_m)$	0.45 (0.051)	0.152 [.112,.226]
$AC1(\Delta c, \Delta d)$	0.637 (0.306)	0.479 [.160,.671]	$E(R_s)$	0.171 (0.049)	0.120 [–]
$\sigma^2\left[(\Delta c_{t+1})^2\right]$	1.0×10^{-6} (4.9×10^{-7})	9.6×10^{-7} $[2.4\times10^{-7}, 2.7\times10^{-6}]$	$E(R_l)$	0.076 (0.020)	0.058 [–]
$\sigma^2\left[(\Delta d_{t+1})^2\right]$	0.0013 (0.0007)	0.0003 $[5.5\times10^{-5}, .0007]$	$E(R_g)$	0.074 (0.023)	0.065 [–]
			$E(R_v)$	0.148 (0.033)	0.109 [–]
J-stat	11.30 (0.010)				

The table reports GMM estimates (asymptotic standard errors in parentheses) of the model parameters defined in Section 1. It also reports the model-implied (95% confidence interval in square brackets obtained through 10,000 simulations) and the historical values (asymptotic standard errors in parentheses) of the mean and volatility of the risk-free rate, price/dividend ratio, and market return, and unconditional moments of the consumption and dividend growth rates. Finally, it reports the J-stat for the overidentifying restrictions along with the associated asymptotic p-value in parentheses.

The model-implied unconditional moments of the aggregate consumption and dividend growth rates in the six-asset system are comparable to those in the two-asset system. However, the six-asset system exacerbates pricing discrepancies on the mean market return and risk-free rate that we previously identified when we estimated the model on the two-asset system. We compute the model-implied mean returns of the "Value," "Growth," "Small" capitalization, and "Large" capitalization portfolios.[4] The annual pricing errors for the "Large" and "Growth" portfolios are small, while the error for the "Small" portfolio is 5.1% and for the "Value" portfolio is 3.9%. The J-stat is 11.30 and the p-value is 1.0%, based on the asymptotic distribution of the J-stat.

3.4 Robustness tests

In Section 3.4.1, we address the robustness of our results to observation error in the price/dividend ratio and risk-free rate. In Section 3.4.2, we examine the robustness of our results to the choice of weighting matrix in the GMM estimation. In Section 3.4.3, we address the robustness of our results to the postwar subperiod.

3.4.1 Observation error in the price/dividend ratio and risk-free rate. A crucial step in observing the latent state variables consists of inverting the system that expresses the risk-free rate and price/dividend ratio as affine functions of the latent state variables. Therefore, the latent state variables are observed with error because both the risk-free rate and price/dividend ratio are observed with error. In particular, our proxy for the one-year real risk-free rate is the deflated one-year nominal risk-free rate. We address the sensitivity of our results to this potential source of error by introducing a third observable, namely the conditional variance of the one-year market return, that is an affine function of the latent state variable σ_t^2 (Equation (19)). We proxy this conditional variance with the sum of squared daily market returns over the previous twelve months.[5] We now have a system of three observables, the risk-free rate, price/dividend ratio, and conditional variance of the one-year market return, as affine functions of the two latent state variables. At each time period, we estimate the values of the latent state variables by a cross-sectional least-squares regression

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4 For the cross-section, the model-implied mean return on portfolio i is computed as $\widehat{E(R_i)} = \frac{1-\widehat{Cov\left(\widehat{M}_t, R_{i,t}\right)}}{\widehat{E(\widehat{M}_t)}}$, where $\widehat{x}$ denotes the estimated value of x and M is the pricing kernel. We compute the model-implied mean returns for the cross-section using this approach because we are unable to simulate returns on these assets without making assumptions about their dividend processes.

5 Andersen, Bollerslev, Diebold, and Labys (2003) and Barndorff-Nielsen and Shephard (2002) show that the sum of squares of high-frequency returns is a highly accurate estimator of the return variance over a discrete time horizon. We do not use the term premia on nominal bonds as the third (or fourth) observable because the conversion of this premia to the term premia on real bonds introduces a maintained hypothesis on the inflation process. We do not use the price/dividend ratio of some other portfolio, for example the value portfolio, as the third observable because this introduces a maintained hypothesis on the dividend growth process of the value portfolio and also involves estimation of the parameters of this process.

of the three observables on their loadings on the latent state variables and proceed as in Section 3.3. The results are reported in Table 4.

The point estimates of the parameters are not significantly different from those in Table 3. The model-implied mean of the market return and the volatility of the risk-free rate are slightly closer to their sample counterparts than the model-implied moments in Table 3. Overall, we find that the introduction of the conditional variance of the one-year market return as a third observable does not significantly enhance the fit of the model.

3.4.2 Sensitivity to the choice of weighting matrix in the GMM estimation. We investigate the robustness of the estimation and tests by replacing the efficient weighting matrix with the identity matrix. Table 5 reports results for the same system of moment restrictions as Table 4, but using the identity weighting matrix. The mean and volatility of the market return and risk-free rate, the volatility of the price-dividend ratio and dividend growth, and the mean of "Small" and "Value" portfolio returns are closer to their sample counterparts than the model-implied moments in Table 4. However, the mean and volatility of consumption growth, the autocorrelation of consumption and dividend growth, and the mean of the price/dividend ratio are further apart from their sample counterparts. Overall, the use of the identity weighting matrix does not unambiguously enhance the fit of the model.

3.4.3 Robustness to the postwar subperiod. Since the period prior to 1947 was one of great economic uncertainty, including the Great Depression, World War II, and structural breaks in the equity premium (Pastor and Stambaugh 2001), the inability of the B-Y model to match certain moments in the data over the full sample period may be due to its poor performance in the prewar period.

We explore this possibility by repeating the estimation and testing of the six-asset system over the postwar subperiod 1947–2009. The results are reported in Table 6 and are worse than those obtained over the full period in Table 3. The model does a better job at matching the unconditional volatility of dividend growth. However, it does a worse job at matching the mean of "Small," "Large," "Growth," and "Value" stocks and the volatility of the market return. The J-stat is 12.38 and has asymptotic p-value less than 1%. Therefore, the inability of the B-Y model to match certain moments in the data cannot be attributed to poor performance in the prewar period.

4. Forecasting Returns and Consumption and Dividend Growth

The B-Y model implies that the conditional expectation of the equity premium is an affine function of the conditional variance, σ_t^2, of the LRR variable (Equation (18)). Bansal, Khatchatrian, and Yaron (2005) show that the

Table 4
Estimation of the B-Y LRR model on the six-asset system using three observables, 1931–2009

Parameter	δ	γ	ψ	μ_c	μ_d	ϕ	φ	ρ_x	ψ_x	σ	υ	σ_w
$Est.^{time-series}$ (Std. Err.)	–	–	–	0.020 (0.003)	0.010 (0.012)	2.06 (0.54)	15.8 (140.8)	0.437 (0.199)	5.20 (47.8)	0.006 (0.049)	0.208 (2.667)	6.0×10^{-5} (0.001)
$Est.^{full\ model}$ (Std. Err.)	0.983 (0.730)	6.69 (23.9)	1.16 (26.4)	0.018 (0.003)	0.011 (0.012)	3.05 (0.765)	6.58 (6.17)	0.711 (0.295)	1.29 (1.58)	0.010 (0.006)	0.425 (1.74)	0.0002 (0.0005)

Moments	Data	Model	Moments	Data	Model
$E(\Delta c_{t+1})$	0.020 (0.003)	0.018 [0.006,0.030]	$E(r_m)$	0.062 (0.019)	0.051 [0.013,0.089]
$\sigma(\Delta c_{t+1})$	0.021 (0.004)	0.021 [0.017,0.033]	$\sigma(r_m)$	0.198 (0.017)	0.127 [0.100,0.179]
$AC1(\Delta c)$	0.449 (0.242)	0.548 [0.210,0.717]	$E(r_f)$	0.006 (0.005)	0.030 [0.021,0.041]
$E(\Delta d_{t+1})$	0.010 (0.013)	0.011 [−0.028,0.051]	$\sigma(r_f)$	0.030 (0.005)	0.017 [0.012,0.026]
$\sigma(\Delta d_{t+1})$	0.117 (0.020)	0.089 [0.075,0.133]	$E(z_m)$	3.38 (0.080)	3.19 [3.12,3.28]
$AC1(\Delta d)$	0.163 (0.136)	0.299 [−0.035,0.526]	$\sigma(z_m)$	0.45 (0.051)	0.147 [0.102,0.211]
$AC(\Delta c, \Delta d)$	0.637 (0.306)	0.570 [0.266,0.743]	$E(R_s)$	0.171 (0.049)	0.119 [–]
$\sigma^2\left[(\Delta c_{t+1})^2\right]$	1.0×10^{-6} (4.9×10^{-7})	2.4×10^{-6} $[3.9\times10^{-7}, 7.1\times10^{-6}]$	$E(R_l)$	0.076 (0.020)	0.059 [–]
$\sigma^2\left[(\Delta d_{t+1})^2\right]$	0.0013 (0.0007)	0.0006 $[9.3\times10^{-5}, 0.0015]$	$E(R_g)$	0.074 (0.023)	0.066 [–]
			$E(R_v)$	0.148 (0.033)	0.108 [–]
J-$stat$	9.94 (0.019)				

The table reports GMM estimates (asymptotic standard errors in parentheses) of the model parameters defined in Section 1. The marketwide price/dividend ratio, risk-free rate, and variance of the market return are used in the extraction of the latent state variables. It also reports the model-implied (95% confidence interval in square brackets obtained through 10,000 simulations) and the historical values (asymptotic standard errors in parentheses) of the mean and volatility of the risk-free rate, price/dividend ratio, and market return, and unconditional moments of the consumption and dividend growth rates. Finally, it reports the J-stat for the overidentifying restrictions along with the associated asymptotic $p$value in parentheses.

Table 5
Estimation of the B-Y model on the six-asset system using the identity weighting matrix, 1931–2009

Parameter	δ	γ	ψ	μ_c	μ_d	ϕ	φ	ρ_x	ψ_x	σ	υ	σ_w
$Est.^{time-series}$	–	–	–	0.020	0.010	2.06	15.8	0.437	5.20	0.006	0.208	6.0×10^{-5}
(*Std. Err.*)				(0.003)	(0.012)	(0.54)	(140.8)	(0.199)	(47.8)	(0.049)	(2.667)	(0.001)
$Est.^{full\ model}$	0.990	4.36	1.84	0.015	0.010	2.15	5.00	0.775	1.42	0.016	0.537	0.0005
(*Std. Err.*)	(1.47)	(21.2)	(47.9)	(0.003)	(0.012)	(1.22)	(11.60)	(0.530)	(2.26)	(0.031)	(50.7)	(0.032)
Moments	*Data*	*Model*				*Moments*	*Data*	*Model*				
$E(\Delta c_{t+1})$	0.020 (0.003)	0.015 [−0.012,0.042]				$E(r_m)$	0.062 (0.019)	0.061 [0.008,0.118]				
$\sigma(\Delta c_{t+1})$	0.021 (0.004)	0.039 [0.030,0.065]				$\sigma(r_m)$	0.198 (0.017)	0.194 [0.149,0.273]				
$AC1(\Delta c)$	0.449 (0.242)	0.647 [0.291,0.790]				$E(r_f)$	0.006 (0.005)	0.006 [−0.015,0.016]				
$E(\Delta d_{t+1})$	0.010 (0.013)	0.010 [−0.050,0.070]				$\sigma(r_f)$	0.030 (0.005)	0.028 [0.018,0.038]				
$\sigma(\Delta d_{t+1})$	0.117 (0.020)	0.109 [0.091,0.176]				$E(z_m)$	3.38 (0.080)	2.97 [2.74,3.07]				
$AC1(\Delta d)$	0.163 (0.136)	0.374 [0.007,0.596]				$\sigma(z_m)$	0.45 (0.051)	0.25 [0.17,0.39]				
$AC(\Delta c, \Delta d)$	0.637 (0.306)	0.635 [0.330,0.792]				$E(R_s)$	0.171 (0.049)	0.150 [–]				
$\sigma^2\left[(\Delta c_{t+1})^2\right]$	1.0×10^{-6} (4.9×10^{-7})	2.2×10^{-5} $[2.5\times10^{-6},7.7\times10^{-5}]$				$E(R_l)$	0.076 (0.020)	0.093 [–]				
$\sigma^2\left[(\Delta d_{t+1})^2\right]$	0.0013 (0.0007)	0.0016 [0.0002,0.0042]				$E(R_g)$	0.074 (0.023)	0.099 [–]				
						$E(R_\upsilon)$	0.148 (0.033)	0.140 [–]				
J-stat	0.135 (<0.01)											

The table reports GMM estimates (asymptotic standard errors in parentheses) of the model parameters defined in Section 1. The marketwide price/dividend ratio, risk-free rate, and conditional variance of the market return are used in the extraction of the latent state variables. It also reports the model-implied (95% confidence interval in square brackets obtained through 10,000 simulations) and the historical values (asymptotic standard errors in parentheses) of the mean and volatility of the risk-free rate, price/dividend ratio, and market return, and unconditional moments of the consumption and dividend growth rates. Finally, it reports the J-stat for the overidentifying restrictions along with the associated asymptotic p-value in parentheses.

Table 6
Estimation of the B-Y LRR model on the six-asset system, 1947–2009

$Parameter$	δ	γ	ψ	μ_c	μ_d	ϕ	φ	ρ_x	ψ_x	σ	υ	σ_w
$Est.^{time-series}$	–	–	–	0.019	0.016	18.7	26.7	0.955	0.243	0.002	0.227	8.7×10^{-6}
$(Std.\ Err.)$				(0.002)	(0.010)	(52.1)	(70.9)	(0.368)	(1.062)	(0.006)	(1362.9)	(0.003)
$Est.^{full\ model}$	0.978	10.2	1.60	0.019	0.017	4.63	10.1	0.838	0.616	0.006	0.297	3.6×10^{-5}
$(Std.\ Err.)$	(0.365)	(14.1)	(14.5)	(0.002)	(0.008)	(1.49)	(5.43)	(0.348)	(0.794)	(0.003)	(8.65)	(0.0002)

$Moments$	$Data$	$Model$	$Moments$	$Data$	$Model$
$E(\Delta c_{t+1})$	0.019	0.019	$E(r_m)$	0.063	0.045
	(0.002)	[0.013,0.025]		(0.020)	[0.011,0.077]
$\sigma(\Delta c_{t+1})$	0.013	0.009	$\sigma(r_m)$	0.176	0.103
	(0.001)	[0.006,0.012]		(0.020)	[0.074,0.136]
$AC1(\Delta c)$	0.454	0.469	$E(r_f)$	0.010	0.033
	(0.138)	[−0.038,0.674]		(0.003)	[0.029,0.037]
$E(\Delta d_{t+1})$	0.016	0.017	$\sigma(r_f)$	0.027	0.004
	(0.010)	[−0.016,0.050]		(0.002)	[0.002,0.006]
$\sigma(\Delta d_{t+1})$	0.070	0.071	$E(z_m)$	3.474	3.57
	(0.009)	[0.049,0.090]		(0.089)	[3.46,3.73]
$AC1(\Delta d)$	0.254	0.175	$\sigma(z_m)$	0.429	0.152
	(0.090)	[−0.216,0.433]		(0.055)	[0.079,0.207]
$AC1(\Delta c, \Delta d)$	0.307	0.342	$E(R_s)$	0.121	0.030
	(0.150)	[−0.089,0.592]		(0.034)	[–]
$\sigma^2\left[(\Delta c_{t+1})^2\right]$	2.1×10^{-7}	1.6×10^{-7}	$E(R_l)$	0.076	0.006
	(4.2×10^{-8})	$[3.8\times10^{-8}, 3.3\times10^{-7}]$		(0.022)	[–]
$\sigma^2\left[(\Delta d_{t+1})^2\right]$	0.0001	0.0001	$E(R_g)$	0.074	0.014
	(5.4×10^{-5})	$[1.6\times10^{-5}, 0.0003]$		(0.025)	[–]
			$E(R_\upsilon)$	0.136	0.032
				(0.024)	[–]
J-$stat$	12.38				
	(0.006)				

The table reports GMM estimates (asymptotic standard errors in parentheses) of the model parameters defined in Section 1. It also reports the model-implied (95% confidence interval in square brackets obtained through 10,000 simulations) and the historical values (asymptotic standard errors in parentheses) of the mean and volatility of the risk-free rate, price/dividend ratio, and market return, and unconditional moments of the consumption and dividend growth rates. Finally, it reports the J-stat for the overidentifying restrictions along with the associated asymptotic p-value in parentheses.

Table 7
Model-implied forecasting regressions

	const.	x_t	σ_t^2	$Adj\text{-}R^2$
Panel A: 1-year, 1931–2009				
$r_{m,t+1} - r_{f,t}$	0.040 (0.027)		60.2 (56.0)	0.002
$r_{m,t+1} - r_{f,t}$	0.074 (0.032)	−0.558 (0.301)	78.5 (56.0)	0.033
Δc_{t+1}	0.016 (0.003)	0.046 (0.031)		0.015
Δc_{t+1}	0.020 (0.003)	0.064 (0.030)	−19.1 (5.52)	0.139
Δd_{t+1}	−0.018 (0.018)	0.399 (0.172)		0.053
Δd_{t+1}	−0.004 (0.018)	0.467 (0.170)	−72.4 (31.7)	0.102
Panel B: 1-year, 1947–2009				
	const.	x_t	σ_t^2	$Adj\text{-}R^2$
$r_{m,t+1} - r_{f,t}$	0.050 (0.027)		18.4 (81.1)	−0.016
$r_{m,t+1} - r_{f,t}$	0.102 (0.032)	−0.848 (0.326)	121.3 (87.0)	0.072
Δc_{t+1}	0.018 (0.003)	0.010 (0.023)		−0.013
Δc_{t+1}	0.019 (0.002)	0.054 (0.023)	−25.8 (6.03)	0.211
Δd_{t+1}	0.015 (0.014)	0.013 (0.122)		−0.016
Δd_{t+1}	0.018 (0.013)	0.153 (0.132)	−82.6 (35.3)	0.053

Panels *A* and *B* report results from model-implied forecasting regressions for the equity premium, and consumption and dividend growth rates over 1931–2009 and 1947–2009, respectively.

conditional volatility of consumption growth predicts valuation ratios. We address the question as to whether the conditional vaiance forecasts the equity premium. We regress the realized annual equity premium, $r_{m,t+1} - r_{f,t}$, on the conditional variance, σ_t^2, over the period 1931–2009. The results are displayed in the first row of Table 7, Panel *A*. The regression coefficient is not statistically significant, and the R^2 is zero.[6] Figure 1 displays the time series of the realized equity premium along with the predicted time series from the above model-implied forecasting regression. Years with NBER recessions in at least two quarters are displayed as shaded columns. The conditional variance exhibits a countercyclical pattern with a correlation coefficient of 0.36 between the time series of the conditional variance and an indicator variable that takes the value of one in a recession year (defined as above) and zero otherwise. The figure shows that the conditional variance does not predict the equity premium, as this conditional variance is flat.[7]

6 Throughout the paper, R^2 refers to the adjusted-R^2.

7 For the results displayed in Figures 1–3 and Table 7, we obtain the time series of the latent state variables from the observed price/dividend ratio, risk-free rate, and the conditional variance of the market return, as in Section 3.4.2. We also obtained the time series of the latent state variables and corresponding figures and tables from the observed price/dividend ratio and risk-free rate, as in Sections 3.2 and 3.3. We chose to display the set of results that cast the model in the best light.

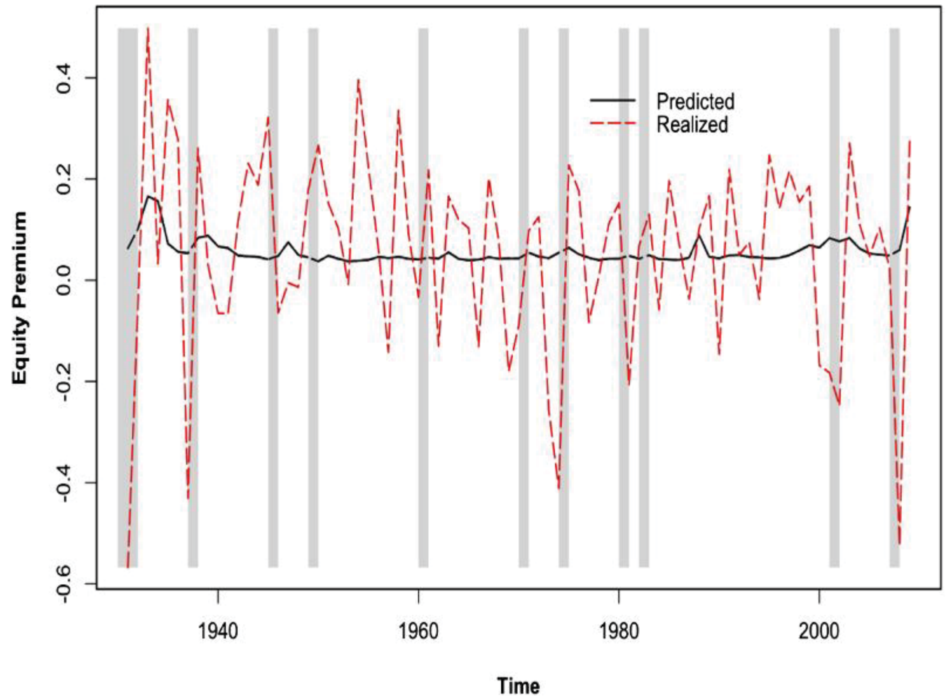

Figure 1
Realized and predicted equity premium, 1931–2009
The figure plots the time series of the realized equity premium (dashed line) along with the premium predicted by the model (solid line). The predicted time series is obtained as the fitted value from a forecasting regression of the realized premium on σ_t^2, the conditional variance of the LRR state variable x_t. The gray shaded areas denote years in which at least two quarters are in NBER-dated recession periods.

Next, we add the LRR variable, x_t, as a second predictor variable in the regression, even though the model implies that the expectation of the equity premium is a function of the state variable σ_t^2 alone. The results are displayed in the second row of Table 7, Panel *A*. The regression coefficient of σ_t^2 remains statistically insignificant, the regression coefficient of x_t is marginally significant, and the R^2 increases to 3.3%. We repeat the regressions over the postwar subperiod 1947–2009 and display the results in the first two rows of Table 7, Panel *B*. In the regression of the equity premium on the conditional variance, the regression coefficient remains insignificant and the R^2 remains zero. In the regression on the conditional variance and the LRR variable, the regression coefficient on σ_t^2 remains statistically insignificant, the coefficient on x_t is strongly statistically significant, and the R^2 increases to 7.2%.

Similar results are obtained at the two-year and five-year frequencies and are reported in Table 8. The B-Y model implies that the two-year and five-year expected equity premia are affine functions of σ_t^2 alone. At the two-year frequency, a forecasting regression of the realized equity premium on σ_t^2 produces a statistically insignificant slope coefficient and R^2 0.6%. When x_t is added as a second predictor variable in the regression, the regression coefficient on σ_t^2 becomes marginally significant, the regression coefficient on x_t is

Table 8
Model-implied forecasting regressions

	const.	x_t	σ_t^2	$Adj\text{-}R^2$
Panel A: Two-year, 1931–2009				
$r_{m,t+1} - r_{f,t}$	0.093 (0.036)		98.65 (81.30)	0.006
$r_{m,t+1} - r_{f,t}$	0.173 (0.041)	−1.296 (0.377)	137.3 (76.91)	0.130
Δc_{t+1}	0.037 (0.005)	0.052 (0.053)		−0.000
Δc_{t+1}	0.039 (0.006)	0.061 (0.053)	−12.58 (10.81)	0.004
Δd_{t+1}	−0.013 (0.027)	0.549 (0.262)		0.042
Δd_{t+1}	−0.011 (0.029)	0.557 (0.267)	−11.58 (54.35)	0.030
Panel B: Five-year, 1931–2009				
	const.	x_t	σ_t^2	$Adj\text{-}R^2$
$r_{m,t+1} - r_{f,t}$	0.287 (0.048)		127.2 (106.9)	0.006
$r_{m,t+1} - r_{f,t}$	0.439 (0.047)	−2.67 (0.46)	210.0 (89.93)	0.314
Δc_{t+1}	0.110 (0.006)	− 0.014 (0.066)		−0.013
Δc_{t+1}	0.107 (0.007)	− 0.027 (0.067)	15.98 (13.00)	−0.006
Δd_{t+1}	0.049 (0.029)	0.576 (0.296)		0.036
Δd_{t+1}	0.246 (0.030)	0.474 (0.292)	126.1 (57.04)	0.085

Panels *A* and *B* report results from model-implied forecasting regressions for the equity premium, and consumption and dividend growth rates over 1931–2009 for two-year and five-year horizons, respectively.

strongly significant, and the R^2 increases by two orders of magnitude to 13.0%. At the five-year frequency, the regression of the equity premium on the conditional variance alone gives R^2 0.6%, while the inclusion of the LRR variable as an additional predictor variable raises the R^2 dramatically to 31.4%.

The overall conclusion is that the conditional variance does not predict the equity premium. This suggests that the dynamics of the conditional variance process in Equation (2) may be misspecified, and the ability of the model to forecast the large time variation in the equity premium may be improved by alternative volatility specifications. Also, the fact that the LRR variable predicts the equity premium, despite the implications of the model to the contrary, suggests that the B-Y model may be enhanced in ways that make the conditional expectation of the equity premium dependent on state variables other than the conditional variance of the LRR variable. Which of these two approaches is more promising is the scope of future research.

The B-Y model also implies that the conditional expectation of the aggregate consumption growth rate is an affine function of the LRR variable (Equation (20)), and the conditional expectation of the aggregate dividend growth rate is an affine function of the LRR variable (Equation (21)). We regress the realized consumption growth on the LRR variable over the period 1931–2009. The results are displayed in the third row of Table 7, Panel *A*. The regression coefficient is not statistically significant, and the R^2 is 1.5%. Figure 2 displays the

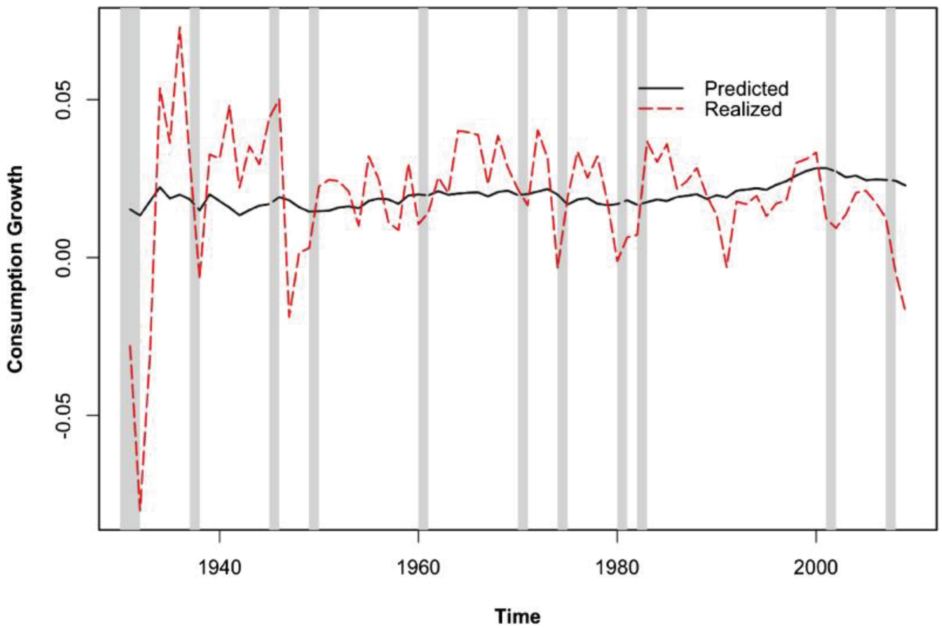

Figure 2
Realized and predicted consumption growth, 1931–2009
The figure plots the time series of the realized consumption growth rate (dashed line) along with the growth rate predicted by the model (solid line). The predicted time series is obtained as the fitted value from a forecasting regression of the realized consumption growth rate on the LRR state variable x_t. The gray shaded areas denote years in which at least two quarters are in NBER-dated recession periods.

time series of the realized consumption growth rate along with the predicted time series from the model-implied forecasting regression. The LRR variable is not very correlated with the business cycle, with a correlation coefficient of only -0.17 between the variable and an indicator variable that takes the value of one in a recession year and zero otherwise.

Next, we add the conditional variance as a second variable in the regression, even though the model does not imply that the expectation of consumption growth is a function of this variable. The results are displayed in the fourth row of Table 7, Panel A. Both coefficients are statistically significant, and the R^2 increases by an order of magnitude from 1.5% to 13.9%. The results in Panel B over the postwar subperiod 1947–2009 make this point even more strongly. The regression of the realized consumption growth rate on the lagged LRR variable, x_t, gives a statistically insignificant coefficient on x_t and negative R^2 (Row 3). Adding the lagged conditional variance, σ_t^2, as a second predictor variable to the regression produces statistically significant coefficients on both variables and 21.1% R^2 (Row 4).

We obtain similar results for the dividend growth rate. Over the period 1931–2009, a forecasting regression of the aggregate dividend growth rate on x_t gives R^2 5.3% (Panel A, Row 5), while the inclusion of the state variable σ_t^2 doubles the R^2 to 10.2% (Panel A, Row 6). Figure 3 displays the time series of the realized dividend growth rate along with the predicted time series from the model-implied forecasting regression of the realized dividend growth rate on

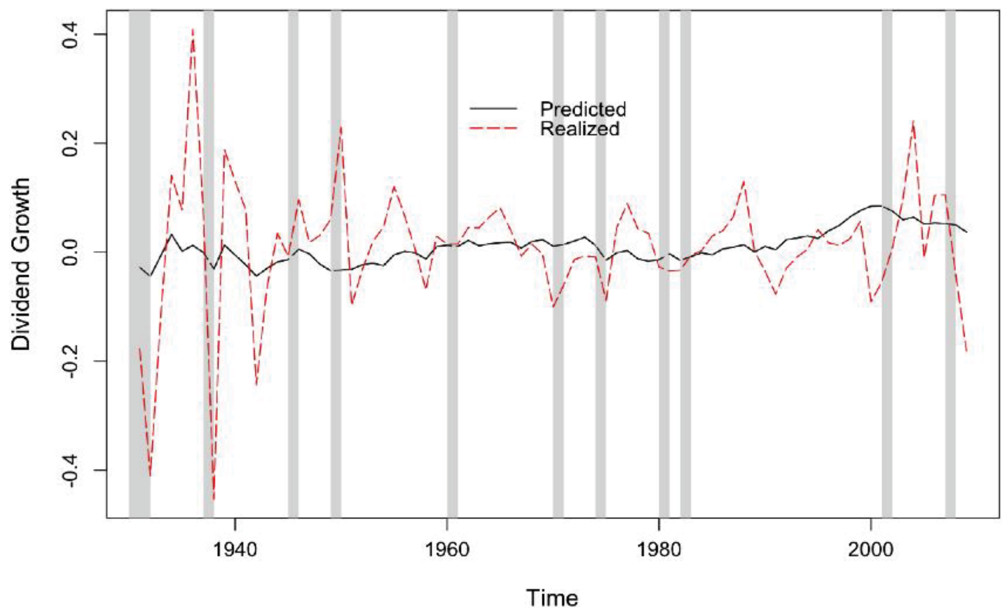

Figure 3
Realized and predicted dividend growth, 1931–2009
The figure plots the time series of the realized dividend growth rate (dashed line) along with the growth rate predicted by the model (solid line). The predicted time series is obtained as the fitted value from a forecasting regression of the realized dividend growth rate on the LRR state variable x_t. The gray shaded areas denote years in which at least two quarters are in NBER-dated recession periods.

the LRR state variable. Over the postwar subperiod 1947–2009, the LRR variable loses its forecasting power for the dividend growth rate with R^2 -1.6% (Panel B, Row 5). The conditional variance, in contrast, predicts the dividend growth with a statistically significant coefficient and R^2 5.3% (Panel B, Row 6).

Overall, the results suggest that the LRR variable does have some predictive power for the consumption and dividend growth rates. However, the conditional variance has strong incremental predictive power for the aggregate consumption and dividend growth rates over and above that contained in the LRR variable, contrary to the implications of the model. Note that, since the expected market return depends on both state variables (Equation (17)), a misspecification of the ex ante risk-free rate could cause both predictors to enter the risk premium regression. Also, the forecasting power of the LRR variable may be an artifact of measurement/estimation error in the extraction of the two state variables and/or time aggregation that might make the innovations to the state variables correlated at the annual frequency. However, the fact that the conditional variance strongly predicts the dividend growth rate in the postwar period even when the LRR variable loses its forecasting power suggests that it is unlikely that our findings are driven entirely by measurement error or time aggregation. The results suggest that the B-Y model may be potentially enhanced in ways that make the conditional expectation of consumption and dividend growth dependent on other state variables in addition to the LRR variable.

5. A Co-integrated Long-run Risks Model

Bansal, Gallant, and Tauchen (2007) consider an extension of the LRR model of B-Y that imposes a co-integrating restriction between the logarithm of the aggregate stock market dividends and consumption. Bansal, Dittmar, and Kiku (2009) point out that this co-integrating relation measures long-run covariance risks in dividends and is important in understanding sources of risk and explaining the equity risk premia across investment horizons.[8] We estimate the log-linearized model and test its implications on the cross-section of returns and on forecasting the equity premium and consumption and dividend growth, using an extension of the methodology introduced in Section 1.1.

5.1 The model and testable implications

The aggregate consumption growth, the LRR variable, and the variance of its innovation are modeled as in Equations (1)–(3). Therefore, the pricing kernel, the log price/consumption ratio, and the risk-free rate are functions of the LRR variable and the variance of its innovation, given by Equations (15), (10), and (12), respectively. The point of departure from the B-Y model is the imposition of a co-integrating restriction between the logarithm of the aggregate stock market dividends and consumption,

$$d_t - c_t = \mu_{dc} + s_t, \tag{22}$$

where the co-integrating residual, s_t, is an $I(0)$ process with the co-integrating coefficient set at one,[9]

$$s_{t+1} = \lambda_{sx} x_t + \rho_s s_t + \psi_s \sigma_t \varepsilon_{s,t+1}. \tag{23}$$

The shocks $\varepsilon_{x,t+1}$, $\varepsilon_{\sigma,t+1}$, $\varepsilon_{c,t+1}$, and $\varepsilon_{s,t+1}$ are assumed to be $i.i.d.\ N(0,1)$ and mutually independent.

From Equation (22), we have

$$\begin{aligned} \Delta d_{t+1} &= \Delta c_{t+1} + \Delta s_{t+1}, \\ &= \mu_c + (1+\lambda_{sx})x_t + (\rho_s - 1)s_t + \sigma_t \varepsilon_{c,t+1} + \psi_s \sigma_t \varepsilon_{s,t+1}, \end{aligned} \tag{24}$$

where the second line follows from Equations (3) and (23).

The model has three state variables, the LRR variable x_t, the variance of its innovation σ_t^2, and the co-integrating residual s_t. Note that the B-Y model obtains as a limiting special case when $\rho_s = 1$. We conjecture that the log

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[8] In a different context, Lettau and Ludvigson (2001) and Menzly, Santos, and Veronesi (2004) apply the co-integrating residual between consumption, labor income, and aggregate stock market dividends to explain the cross-section of returns.

[9] Bansal, Gallant, and Tauchen (2007) perform a heteroscedasticity-robust augmented Dickey-Fuller test for a unit root in $d_t - c_t$, and the results provide strong evidence for a co-integrating relationship between the variables with a coefficient equal to unity.

price/dividend ratio is an affine function of the LRR variable, the variance of its innovation, and the co-integrating residual. In Appendix A.3.1, we verify this conjecture and explicitly solve for the coefficients. The co-integrating residual is observable as the demeaned difference between the log aggregate dividend and consumption levels (Equation (22)). We invert the equations for the equilibrium risk-free rate and marketwide price/dividend ratio and express the unobservable state variables, x_t and σ_t^2, in terms of the observables, $z_{m,t}$, $r_{f,t}$, and s_t (see Appendix A.3.2 for details). Finally, we express the pricing kernel as an affine function of $z_{m,t}$, $r_{f,t}$, and s_t; their lags; and consumption growth.

5.2 Empirical evidence on the co-integrated model

We estimate the preference parameters and the parameters of the time-series processes of aggregate consumption and dividend growth over 1931–2009 by GMM from the joint system of Euler equations and the restrictions on the unconditional moments of consumption and dividend growth imposed by the time-series specification of the model. The asset menu consists of the market portfolio, risk-free rate, and portfolios of “Small” capitalization, “Large” capitalization, “Growth,” and “Value” stocks. The Euler equations for the six assets give six moment restrictions. To this set of pricing restrictions, we add the following seven time-series moment restrictions: the unconditional mean, variance, and first- and second-order autocorrelations of consumption growth, the variance and first-order autocorrelation of dividend growth, and the correlation between consumption and dividend growth (see Appendix A.4 for expressions for these moments). Thus, we have a total of thirteen moment conditions. The total number of parameters to be estimated is twelve (nine time-series parameters and three preference parameters). We estimate the parameters with GMM using the efficient weighting matrix and test the model with the overidentifying restriction.

The results are reported in Table 9. The point estimate of the parameter ρ_s, which determines the persistence of the co-integrating residual, s_t, is 0.90 and is statistically indistinguishable from unity. Therefore, the data cannot distinguish the co-integrated model from the B-Y model, which obtains as a limiting special case when $\rho_s = 1$. This explains why the conclusions drawn from Table 9 are similar to our earlier conclusions from Table 3. The persistence parameter of the LRR variable is much higher at 0.96, compared with the value of 0.44 estimated from the time-series model alone in the first row of Table 3. Therefore, the co-integrated model, like the B-Y model, requires much higher persistence of consumption growth to explain the cross-section of returns than the persistence estimated from the time series of consumption growth alone. The model does a fair job at matching the unconditional moments of the consumption and dividend growth rates. However, like the B-Y model, it implies a higher level of the risk-free rate than that observed in the data (2.8% versus

Table 9
Estimation of the co-integrated model on the six-asset system, 1931–2009

Parameter	δ	γ	ψ	μ_c	ρ_x	ψ_x	σ	υ	σ_w	λ_{sx}	ρ_s	ψ_s
Estimate	0.98 (0.892)	6.0 (67.2)	1.2 (101.3)	0.011 (0.003)	0.96 (0.613)	0.189 (1.538)	0.014 (0.003)	0.983 (3.324)	9.3×10^{-6} (0.013)	9.305 (52.3)	0.90 (2.35)	4.798 (10.1)

Moments	*Data*	*Model*	*Moments*	*Data*	*Model*
$E(\Delta c_{t+1})$	0.020 (0.003)	0.011 [0.0002,0.022]	$E(r_m)$	0.062 (0.019)	0.045 [0.017,0.074]
$\sigma(\Delta c_{t+1})$	0.021 (0.004)	0.017 [0.011,0.019]	$\sigma(r_m)$	0.198 (0.017)	0.093 [0.072,0.113]
$AC1(\Delta c)$	0.449 (0.242)	0.300 [−0.12,0.44]	$E(r_f)$	0.006 (0.005)	0.028 [0.019,0.036]
$E(\Delta d_{t+1})$	0.010 (0.013)	0.011 [−0.016,0.038]	$\sigma(r_f)$	0.030 (0.005)	0.008 [0.003,0.009]
$\sigma(\Delta d_{t+1})$	0.117 (0.020)	0.087 [0.064,0.106]	$E(z_m)$	3.38 (0.080)	3.37 [3.03,3.69]
$AC1(\Delta d)$	0.163 (0.136)	0.295 [−0.04,0.51]	$\sigma(z_m)$	0.45 (0.051)	0.37 [0.15,0.41]
$AC(\Delta c, \Delta d)$	0.637 (0.306)	0.352 [0.11,0.54]	$E(R_s)$	0.171 (0.049)	0.163 [−]
			$E(R_l)$	0.076 (0.020)	0.107 [−]
			$E(R_g)$	0.074 (0.023)	0.115 [−]
			$E(R_v)$	0.148 (0.033)	0.148 [−]
J-stat	17.2 (0.000)				

The table reports GMM estimates (asymptotic standard errors in parentheses) of the model parameters defined in Section 5. It also reports the model-implied (95% confidence interval in square brackets obtained through 10,000 simulations) and the historical values (asymptotic standard errors in parentheses) of the mean and volatility of the risk-free rate, price/dividend ratio, and market return, and unconditional moments of the consumption and dividend growth rates. Finally, it reports the J-stat for the overidentifying restrictions along with the associated asymptotic p-value in parentheses.

0.6%), lower volatility of the risk-free rate (0.8% versus 3.0%), and lower volatility of the market return (9.3% versus 19.8%). The model performs better in matching the volatility of the price/dividend ratio compared with the B-Y model. The annual pricing errors for the "Small" capitalization and "Value" portfolios are better than those obtained for the B-Y model, but the pricing error for the "Growth" portfolio is worse. Finally, the GMM overidentifying restrictions test rejects this model with J-stat 17.2 and asymptotic p-value less than 1%.

We next examine the forecasting power of the model-implied state variables for the equity premium and the aggregate consumption and dividend growth rates. The co-integrated model implies that the conditional expectation of the equity premium is an affine function of the conditional variance, σ_t^2, of the LRR variable (see Appendix A.3.3 for derivation). We regress the realized equity premium, $r_{m,t+1} - r_{f,t}$, on the conditional variance, σ_t^2, over the period 1931–2009. The results are displayed in the first row of Table 10, Panel A. The regression coefficient is not statistically significant, and the R^2 is -1.1%. Next, we add the LRR variable, x_t, as a second predictor variable in the regression, even though the model implies that the expected equity premium is a function of the state variable σ_t^2 alone. The results are displayed in the second row of Table 10, Panel A. The regression coefficients are statistically insignificant, and the R^2 is negative. Row 3 shows that inclusion of the co-integrating residual, s_t, as a third state variable makes all three regression coefficients statistically indistinguishable from zero and the R^2 is still negative. We repeat the regressions over the postwar subperiod 1947–2009 and display the results in the first three rows of Table 10, Panel B, with similar results. The overall conclusion is that σ_t^2 does not predict the equity premium, contrary to the predictions of the model. This conclusion is similar to that obtained for the B-Y model. The fact that the other state variables, namely the LRR variable and the co-integrating residual, also do not have any forecasting power for the equity premium suggests that this class of models is missing important state variables that drive the dynamics of the equity premium.

The co-integrated model, like the B-Y model, also implies that the conditional expectation of the aggregate consumption growth rate is an affine function of the LRR variable (Equation (20)). We regress the realized consumption growth on the LRR variable over the period 1931–2009. The results are displayed in the fourth row of Table 10, Panel A. The regression coefficient has the wrong sign, and the R^2 is 2.6%. Next, we add the conditional variance as a second variable in the regression, even though the model does not imply that the expected consumption growth is a function of this variable. The results are displayed in the fifth row of Table 10, Panel A. The coefficient on σ_t^2 is strongly statistically significant, and the R^2 rises by an order of magnitude from 2.6% to 27.8%. Row 6 shows that inclusion of the co-integrating residual, s_t, as a third state variable does not change the outcome. The results show that the conditional variance has strong predictive power for the aggregate consumption

Table 10
Co-integrated model-implied forecasting regressions

	$const.$	x_t	σ_t^2	s_t	$Adj\text{-}R^2$
Panel A: One-year, 1931–2009					
$r_{m,t+1} - r_{f,t}$	0.062 (0.027)		−46.88 (77.17)		−0.011
$r_{m,t+1} - r_{f,t}$	0.051 (0.040)	−0.270 (0.591)	−19.76 (69.16)		−0.017
$r_{m,t+1} - r_{f,t}$	0.047 (0.040)	−0.328 (0.594)	−8.52 (70.26)	−0.100 (0.107)	−0.019
Δc_{t+1}	0.017 (0.003)	−0.085 (0.048)			0.026
Δc_{t+1}	0.030 (0.003)	0.080 (0.052)	−32.12 (6.081)		0.278
Δc_{t+1}	0.030 (0.004)	0.078 (0.052)	−31.62 (6.205)	−0.004 (0.009)	0.271
Δd_{t+1}	0.002 (0.016)	−0.226 (0.264)		−0.163 (0.059)	0.076
Δd_{t+1}	0.052 (0.021)	0.400 (0.313)	−122.1 (36.95)	−0.131 (0.056)	0.183
Panel B: One-year, 1947–2009					
	$const.$	x_t	σ_t^2	s_t	$Adj\text{-}R^2$
$r_{m,t+1} - r_{f,t}$	0.065 (0.026)		−38.76 (55.02)		−0.007
$r_{m,t+1} - r_{f,t}$	0.064 (0.039)	0.045 (0.635)	−50.30 (91.80)		−0.027
$r_{m,t+1} - r_{f,t}$	0.063 (0.039)	0.027 (0.634)	−45.01 (94.89)	0.031 (0.122)	−0.043
Δc_{t+1}	0.023 (0.002)	0.128 (0.037)			0.153
Δc_{t+1}	0.026 (0.003)	0.164 (0.043)	− 9.752 (6.185)		0.174
Δc_{t+1}	0.025 (0.003)	0.157 (0.043)	− 7.736 (6.286)	0.011 (0.008)	0.188
Δd_{t+1}	0.018 (0.011)	0.062 (0.215)		−0.036 (0.047)	−0.022
Δd_{t+1}	0.027 (0.016)	0.174 (0.256)	−30.64 (37.79)	−0.045 (0.048)	−0.028

Panels A and B report results from the co-integrated model-implied forecasting regressions for the equity premium, and consumption and dividend growth rates over 1931–2009 and 1947–2009, respectively.

growth rate, contrary to the implications of the model. The results in Panel B over the postwar subperiod 1947–2009 show that the LRR variable performs well at forecasting the consumption growth rate over this period with the conditional variance and the co-integrating residual not having much incremental forecasting power.

Finally, the co-integrated model, unlike the B-Y model, implies that the conditional expectation of the aggregate dividend growth rate is an affine function of the LRR variable and the co-integrating residual (Equation (24)). We regress the realized dividend growth on x_t and s_t over the period 1931–2009. The results are displayed in Row 7 of Table 10, Panel A. The regression coefficient of the co-integrating residual is statistically significant, that of the LRR variable is not, and the R^2 is 7.6%. Next, we add σ_t^2 as a third variable in the regression, contrary to the implications of the model. The results are displayed in Row 8. The coefficient on σ_t^2 is strongly statistically significant, and the R^2 more than doubles to 18.3%. This shows that the conditional variance has strong predictive power for the aggregate dividend growth rate, contrary to the implications of the model. The results in Panel *B* over the postwar subperiod 1947–2009 show that none of the state variables have statistically significant coefficients and in both cases the R^2 is negative.

The co-integrated model generalizes the LRR model of B-Y by introducing the difference between the log dividend and consumption levels as a third state variable. The combined evidence from the estimation, pricing tests, and forecasting regressions suggests that the problems identified with the model of B-Y remain to be resolved.

6. Concluding Remarks

We presented a novel methodology for estimating and testing the class of long-run risks models and related models that contain latent state variables. We illustrated the methodology by estimating and testing the long-run risks model of Bansal and Yaron (2004) and its co-integrated extension in Bansal, Gallant, and Tauchen (2007), and we provided insights for building the next generation of such models. The results are summarized in the introduction. Recent studies by Ferson, Nallareddy, and Xie (2011), Ghosh and Constantinides (2011), and Jagannathan and Marakani (2010) are already building on this methodology.

The main difficulty in assessing the empirical plausibility of such models is their reliance on latent state variables. The methodology is based on the insight that the model yields expressions of various observable quantities, such as the marketwide price/dividend ratio and risk-free rate, as functions of the latent state variables and the model parameters. These functions may be inverted to express the latent state variables as known functions of the observables and the model parameters. The procedure bypasses the need to filter the state variables and, more importantly, bypasses the need to spell out the information set over which consumers filter the state variables.

The latent state variables may be readily related to the time series of financial and macroeconomic variables. The stochastic discount factor and the Euler equations of consumption are expressed in terms of these observables. The model may be estimated and tested with one-step procedures, such as GMM, on the joint system of the Euler equations and the unconditional moments of observables.

Finally, the methodology yields novel testable implications on predictability. For example, whereas the B-Y model implies that the conditional mean of the equity premium is a function of the price/dividend ratio and risk-free rate (and, equivalently, a function of the LRR variable and its conditional variance), closer examination reveals that the model has the sharper implication that the conditional mean of the equity premium is a function of the conditional variance of the LRR variable but not of the LRR variable itself. This sharper implication is readily testable with the methodology that reveals the latent state variables.

A. Appendix

A.1 Estimation of time-series parameters of the B-Y model

The decision interval of the agent is assumed to be annual. We estimate the model at the annual frequency, such that its annual growth rates of consumption and dividends match salient features of observed annual consumption and dividend data. There are nine parameters to be estimated—μ_c, μ_d, ϕ, φ, ρ_x, ψ_x, σ, υ, and σ_w.

From the specification of the consumption growth process, we have

$$E\left(\Delta c_{t+1}\right) = \mu_c. \tag{25}$$

We also have

$$\begin{aligned} Var\left(\Delta c_{t+1}\right) &= Var\left(x_t\right) + Var\left(\sigma_t \varepsilon_{c,t+1}\right) + 2Cov(x_t, \sigma_t \varepsilon_{c,t+1}) \\ &= Var\left(x_t\right) + \sigma^2 + 0 \\ &= \frac{\psi_x^2 \sigma^2}{1-\rho_x^2} + \sigma^2 \end{aligned} \tag{26}$$

and

$$Cov(\Delta c_{t+1}, \Delta c_{t+2}) = \rho_x \frac{\psi_x^2 \sigma^2}{1-\rho_x^2}. \tag{27}$$

From the specification of the dividend process, we have

$$E\left(\Delta d_{t+1}\right) = \mu_d \tag{28}$$

$$Var\left(\Delta d_{t+1}\right) = \phi^2 \frac{\psi_x^2 \sigma^2}{1-\rho_x^2} + \sigma^2 \varphi^2 \tag{29}$$

$$Cov(\Delta d_{t+1}, \Delta d_{t+2}) = \phi^2 \rho_x \frac{\psi_x^2 \sigma^2}{1-\rho_x^2}. \tag{30}$$

Also, from the consumption and dividend growth processes,

$$Cov(\Delta c_{t+1}, \Delta d_{t+1}) = \phi \frac{\psi_x^2 \sigma^2}{1-\rho_x^2}. \tag{31}$$

Finally, we have

$$Var\left(\left(\Delta c_{t+1}\right)^2\right) = E\left[Var_t\left(\left(\Delta c_{t+1}\right)^2\right)\right] + Var\left[E_t\left(\left(\Delta c_{t+1}\right)^2\right)\right]. \tag{32}$$

Now,

$$\left(\Delta c_{t+1}\right)^2 = \mu_c^2 + x_t^2 + \sigma_t^2 \varepsilon_{c,t+1}^2 + 2\mu_c x_t + 2x_t \sigma_t \varepsilon_{c,t+1} + 2\mu_c \sigma_t \varepsilon_{c,t+1}. \tag{33}$$

Hence,

$$E_t\left(\left(\Delta c_{t+1}\right)^2\right) = \mu_c^2 + x_t^2 + \sigma_t^2 + 2\mu_c x_t$$

$$\begin{aligned} Var\left[E_t\left(\left(\Delta c_{t+1}\right)^2\right)\right] &= Var(x_t^2) + Var(\sigma_t^2) + 4\mu_c^2 Var(x_t) + 4\mu_c Cov(x_t, x_t^2) \\ &\quad + 2Cov(x_t^2, \sigma_t^2) + 4\mu_c Cov(x_t, \sigma_t^2). \end{aligned} \tag{34}$$

Now, $Var(\sigma_t^2) = \frac{\sigma_w^2}{1-\upsilon^2}$, $Cov(x_t, \sigma_t^2) = 0$, $Cov(x_t^2, \sigma_t^2) = \frac{\psi_x^2 \sigma_w^2 \upsilon}{(1-\upsilon^2)(1-\upsilon\rho_x^2)}$, $Cov(x_t, x_t^2) = 0$, and

$$Var(x_t^2) = \frac{3\psi_x^4 \sigma_w^2 (1+\upsilon\rho_x^2)}{(1-\rho_x^4)(1-\upsilon^2)(1-\upsilon\rho_x^2)} + \frac{1}{1-\rho_x^4}\left[2\sigma^4 + \frac{4\rho_x^2\psi_x^4\sigma^4}{(1-\rho_x^2)}\right].$$

Substituting the above expressions into Equation (34), we have

$$Var\left[E_t\left(\left(\Delta c_{t+1}\right)^2\right)\right] = \frac{3\psi_x^4 \sigma_w^2 (1+\upsilon\rho_x^2)}{(1-\rho_x^4)(1-\upsilon^2)(1-\upsilon\rho_x^2)} + \frac{1}{1-\rho_x^4}\left[2\sigma^4 + \frac{4\rho_x^2\psi_x^4\sigma^4}{(1-\rho_x^2)}\right] + \frac{\sigma_w^2}{1-\upsilon^2} + 4\mu_c^2\frac{\psi_x^2\sigma^2}{1-\rho_x^2} + \frac{2\psi_x^2\sigma_w^2\upsilon}{(1-\upsilon^2)(1-\upsilon\rho_x^2)}. \quad (35)$$

Also, from Equation (33),

$$Var_t\left(\left(\Delta c_{t+1}\right)^2\right) = 2\sigma_t^4 + 4x_t^2\sigma_t^2 + 4\mu_c^2\sigma_t^2 + 8\mu_c x_t \sigma_t^2.$$

Hence,

$$E\left[Var_t\left(\left(\Delta c_{t+1}\right)^2\right)\right] = 2\frac{\sigma_w^2}{1-\upsilon^2} + 2\sigma^4 + \frac{4\psi_x^2\sigma_w^2\upsilon}{(1-\upsilon^2)(1-\upsilon\rho_x^2)} + \frac{4\psi_x^2\sigma^4}{1-\rho_x^2} + 4\mu_c^2\sigma^2. \quad (36)$$

Substituting Equations (35) and (36) into Equation (32), we have

$$Var\left(\left(\Delta c_{t+1}\right)^2\right) = \frac{3\psi_x^4 \sigma_w^2 (1+\upsilon\rho_x^2)}{(1-\rho_x^4)(1-\upsilon^2)(1-\upsilon\rho_x^2)} + \frac{1}{1-\rho_x^4}\left[2\sigma^4 + \frac{4\rho_x^2\psi_x^4\sigma^4}{(1-\rho_x^2)}\right] + \frac{3\sigma_w^2}{1-\upsilon^2} + 4\mu_c^2\frac{\psi_x^2\sigma^2}{1-\rho_x^2} + \frac{6\psi_x^2\sigma_w^2\upsilon}{(1-\upsilon^2)(1-\upsilon\rho_x^2)} + \frac{4\psi_x^2\sigma^4}{1-\rho_x^2} + 2\sigma^4 + 4\mu_c^2\sigma^2. \quad (37)$$

Similar calculations yield

$$Var\left[E_t\left(\left(\Delta d_{t+1}\right)^2\right)\right] = \phi^4\left[\frac{3\psi_x^4 \sigma_w^2 (1+\upsilon\rho_x^2)}{(1-\rho_x^4)(1-\upsilon^2)(1-\upsilon\rho_x^2)} + \frac{1}{1-\rho_x^4}\left(2\sigma^4 + \frac{4\rho_x^2\psi_x^4\sigma^4}{(1-\rho_x^2)}\right)\right] + \frac{\sigma_w^2}{1-\upsilon^2}\varphi^4 + 4\mu_c^2\frac{\psi_x^2\sigma^2}{1-\rho_x^2}\phi^2 + \frac{2\psi_x^2\sigma_w^2\upsilon}{(1-\upsilon^2)(1-\upsilon\rho_x^2)}\phi^2\varphi^2$$

$$E\left[Var_t\left(\left(\Delta d_{t+1}\right)^2\right)\right] = \left[2\frac{\sigma_w^2}{1-\upsilon^2} + 2\sigma^4\right]\varphi^4 + \left[\frac{4\psi_x^2\sigma_w^2\upsilon}{(1-\upsilon^2)(1-\upsilon\rho_x^2)} + \frac{4\psi_x^2\sigma^4}{1-\rho_x^2}\right]\phi^2\varphi^2 + 4\mu_d^2\varphi^2\sigma^2.$$

Hence, we have

$$Var\left(\left(\Delta d_{t+1}\right)^2\right) = \phi^4\left[\frac{3\psi_x^4 \sigma_w^2 (1+\upsilon\rho_x^2)}{(1-\rho_x^4)(1-\upsilon^2)(1-\upsilon\rho_x^2)} + \frac{1}{1-\rho_x^4}\left(2\sigma^4 + \frac{4\rho_x^2\psi_x^4\sigma^4}{(1-\rho_x^2)}\right)\right] + \frac{3\sigma_w^2}{1-\upsilon^2}\varphi^4 + 4\mu_c^2\frac{\psi_x^2\sigma^2}{1-\rho_x^2}\phi^2 + \frac{6\psi_x^2\sigma_w^2\upsilon}{(1-\upsilon^2)(1-\upsilon\rho_x^2)}\phi^2\varphi^2 + \frac{4\psi_x^2\sigma^4}{1-\rho_x^2}\phi^2\varphi^2 + 2\sigma^4\varphi^4 + 4\mu_d^2\varphi^2\sigma^2. \quad (38)$$

Equations (25)–(31), (37), and (38) give nine moment restrictions in the nine time-series parameters.

A.2 Details of Estimation Methodology for the B-Y Model

A.2.1 Expressions for A_0, A_1, A_2, $A_{0,m}$, $A_{1,m}$, and $A_{2,m}$. Bansal and Yaron (2004) show that z_t and $z_{m,t}$ are affine functions of the state variables x_t and σ_t^2,

$$z_t = A_0 + A_1 x_t + A_2 \sigma_t^2,$$
$$z_{m,t} = A_{0,m} + A_{1,m} x_t + A_{2,m} \sigma_t^2,$$

where

$$A_1 = \frac{1 - \frac{1}{\psi}}{1 - \kappa_1 \rho_x}$$
$$A_2 = \frac{0.5\left[\left(-\frac{\theta}{\psi} + \theta\right)^2 + (\theta \kappa_1 A_1 \psi_x)^2\right]}{\theta\,(1 - \kappa_1 \upsilon)}$$
$$A_0 = \frac{log(\delta) + \left(1 - \frac{1}{\psi}\right)\mu_c + \kappa_0 + \kappa_1 A_2 \sigma^2 (1 - \upsilon) + 0.5\theta \kappa_1^2 A_2^2 \sigma_w^2}{1 - \kappa_1}$$

$$A_{1,m} = \frac{\phi - \frac{1}{\psi}}{1 - \kappa_{1,m}\rho_x}$$
$$A_{2,m} = \frac{(1-\theta) A_2\,(1 - \kappa_1 \upsilon) + 0.5\left[\gamma^2 + \varphi^2 + \left((\theta - 1)\,\kappa_1 A_1 + \kappa_{1,m} A_{1,m}\right)^2 \psi_x^2\right]}{1 - \kappa_{1,m}\upsilon}$$
$$A_{0,m} = \frac{\theta \log(\delta) + \left(-\frac{\theta}{\psi} + \theta - 1\right)\mu_c + (\theta - 1)\,\kappa_0 + (\theta - 1)\,(\kappa_1 - 1)\,A_0 + (\theta - 1)\,\kappa_1 A_2 \sigma^2 (1 - \upsilon)}{1 - \kappa_{1,m}}$$
$$+ \frac{\kappa_{0,m} + \mu_d + \kappa_{1,m} A_{2,m} \sigma^2 (1 - \upsilon) + 0.5\left[(\theta - 1)\,\kappa_1 A_2 + \kappa_{1,m} A_{2,m}\right]^2 \sigma_w^2}{1 - \kappa_{1,m}}.$$

Finally, we express the linearization parameters κ_0 and κ_1 in terms of the preference and time-series parameters through the restriction that the long-run mean of the log price/consumption ratio, $\bar{z}$, that defines κ_0 and κ_1 should equal the unconditional expectation of z_t implied by Equation (10); and we express the linearization parameters $\kappa_{0,m}$ and $\kappa_{1,m}$ in terms of the preference and time-series parameters through the restriction that the long-run mean of the log price/dividend ratio, $\bar{z}_m$, that defines $\kappa_{0,m}$ and $\kappa_{1,m}$ should equal the unconditional expectation of $z_{m,t}$ implied by Equation (11).

A.2.2 Risk-free rate. To derive the expression for the risk-free rate, note that

$$E_t\left[\exp\left(\theta \log \delta - \frac{\theta}{\psi}\Delta c_{t+1} + (\theta - 1) r_{c,t+1} + r_{f,t}\right)\right] = 1.$$

Hence,

$$\begin{aligned}
\exp\left(-r_{f,t}\right) &= E_t\left[\exp\left(\theta \log \delta - \frac{\theta}{\psi}\Delta c_{t+1} + (\theta - 1) r_{c,t+1}\right)\right] \\
&= \exp\left(\theta \log \delta - \frac{\theta}{\psi}\mu_c - \frac{\theta}{\psi} x_t + (\theta - 1)\kappa_0 + (\theta - 1)\kappa_1 A_0\right. \\
&\quad + (\theta - 1)\kappa_1 A_1 \rho_x x_t + (\theta - 1)\kappa_1 A_2 (1 - \upsilon)\sigma^2 + (\theta - 1)\kappa_1 A_2 \upsilon \sigma_t^2 \\
&\quad - (\theta - 1) A_0 - (\theta - 1) A_1 x_t - (\theta - 1) A_2 \sigma_t^2 + (\theta - 1)\mu_c + (\theta - 1) x_t \\
&\quad \left. + 0.5\left[\left(-\frac{\theta}{\psi} + \theta - 1\right)^2 \sigma_t^2 + (\theta - 1)^2 \kappa_1^2 A_1^2 \psi_x^2 \sigma_t^2 + (\theta - 1)^2 \kappa_1^2 A_2^2 \sigma_w^2\right]\right).
\end{aligned}$$

Therefore, the risk-free rate is

$$
\begin{aligned}
r_{f,t} =& -\theta \log \delta - \left(-\frac{\theta}{\psi} + \theta - 1\right)\mu_c - (\theta-1)\kappa_0 - (\theta-1)(\kappa_1 - 1)A_0 - (\theta-1)\kappa_1 A_2(1-\upsilon)\sigma^2 \\
& - 0.5(\theta-1)^2\kappa_1^2 A_2^2 \sigma_w^2 - \left[(-\frac{\theta}{\psi} + \theta - 1) + (\theta-1)(\kappa_1\rho_x - 1)A_1\right] x_t \\
& - \left[(\theta-1)(\kappa_1\upsilon - 1)A_2 + 0.5\left(\left(-\frac{\theta}{\psi} + \theta - 1\right)^2 + (\theta-1)^2\kappa_1^2 A_1^2 \psi_x^2\right)\right]\sigma_t^2 \\
=& A_{0,f} + A_{1,f}x_t + A_{2,f}\sigma_t^2,
\end{aligned}
$$

where

$$
\begin{aligned}
A_{0,f} =& -\theta \log \delta - \left(-\frac{\theta}{\psi} + \theta - 1\right)\mu_c - (\theta-1)\kappa_0 - (\theta-1)(\kappa_1 - 1)A_0 - (\theta-1)\kappa_1 A_2(1-\upsilon)\sigma^2 \\
& - 0.5(\theta-1)^2\kappa_1^2 A_2^2 \sigma_w^2 \\
A_{1,f} =& -\left[(-\frac{\theta}{\psi} + \theta - 1) + (\theta-1)(\kappa_1\rho_x - 1)A_1\right] \\
A_{2,f} =& -\left[(\theta-1)(\kappa_1\upsilon - 1)A_2 + 0.5\left(\left(-\frac{\theta}{\psi} + \theta - 1\right)^2 + (\theta-1)^2\kappa_1^2 A_1^2 \psi_x^2\right)\right].
\end{aligned}
$$

A.2.3 Latent state variables in terms of observable variables. The model implies

$$
\begin{aligned}
z_{m,t} &= A_{0,m} + A_{1,m}x_t + A_{2,m}\sigma_t^2, \\
r_{f,t} &= A_{0,f} + A_{1,f}x_t + A_{2,f}\sigma_t^2.
\end{aligned}
$$

These equations may be inverted to express the state variables in terms of the observables,

$$x_t = \alpha_0 + \alpha_1 r_{f,t} + \alpha_2 z_{m,t},$$

where

$$
\begin{aligned}
\alpha_0 &= \frac{A_{2,m}A_{0,f} - A_{0,m}A_{2,f}}{A_{1,m}A_{2,f} - A_{2,m}A_{1,f}}, \\
\alpha_1 &= \frac{-A_{2,m}}{A_{1,m}A_{2,f} - A_{2,m}A_{1,f}}, \\
\alpha_2 &= \frac{A_{2,f}}{A_{1,m}A_{2,f} - A_{2,m}A_{1,f}},
\end{aligned}
$$

and

$$\sigma_t^2 = \beta_0 + \beta_1 r_{f,t} + \beta_2 z_{m,t},$$

where

$$
\begin{aligned}
\beta_0 &= \frac{A_{0,m}A_{1,f} - A_{1,m}A_{0,f}}{A_{1,m}A_{2,f} - A_{2,m}A_{1,f}}, \\
\beta_1 &= \frac{A_{1,m}}{A_{1,m}A_{2,f} - A_{2,m}A_{1,f}}, \\
\beta_2 &= \frac{-A_{1,f}}{A_{1,m}A_{2,f} - A_{2,m}A_{1,f}}.
\end{aligned}
$$

A.2.4 The pricing kernel in terms of observables. The pricing kernel is given by (15),

$$m_{t+1} = \left(\theta \log \delta + (\theta - 1)\left[\kappa_0 + (\kappa_1 - 1) A_0\right]\right) + \left(-\frac{\theta}{\psi} + (\theta - 1)\right) \Delta c_{t+1}$$
$$+ (\theta - 1)\kappa_1 A_1 x_{t+1} + (\theta - 1)\kappa_1 A_2 \sigma_{t+1}^2 - (\theta - 1) A_1 x_t - (\theta - 1) A_2 \sigma_t^2.$$

Substituting the expressions for x_t and σ_t^2 from Section A.2.3 into the pricing kernel, we have

$$m_{t+1} = c_1 + c_2 \Delta c_{t+1} + c_3 \left(r_{f,t+1} - \frac{1}{\kappa_1} r_{f,t}\right) + c_4 \left(z_{m,t+1} - \frac{1}{\kappa_1} z_{m,t}\right),$$

where

$$c_1 = \theta \log \delta + (\theta - 1)[\kappa_0 + (\kappa_1 - 1)\,(A_0 + A_1 \alpha_0 + A_2 \beta_0)]$$
$$c_2 = -\frac{\theta}{\psi} + (\theta - 1)$$
$$c_3 = (\theta - 1)\kappa_1 [A_1 \alpha_1 + A_2 \beta_1]$$
$$c_4 = (\theta - 1)\kappa_1 [A_1 \alpha_2 + A_2 \beta_2].$$

A.3 Estimation methodology for the co-integrated model

The model is given by the equations

$$\begin{aligned}
\Delta c_{t+1} &= \mu_c + x_t + \sigma_t \varepsilon_{c,t+1}, \\
x_{t+1} &= \rho_x x_t + \psi_x \sigma_t \varepsilon_{x,t+1}, \\
\sigma_{t+1}^2 &= (1 - \upsilon)\,\sigma^2 + \upsilon \sigma_t^2 + \sigma_w \varepsilon_{\sigma,t+1}, \\
d_t - c_t &= \mu_{dc} + s_t, \\
s_{t+1} &= \lambda_{sx} x_t + \rho_s s_t + \psi_s \sigma_t \varepsilon_{s,t+1}, \\
\Delta d_{t+1} &= \mu_c + (1 + \lambda_{sx}) x_t + (\rho_s - 1) s_t + \sigma_t \varepsilon_{c,t+1} + \psi_s \sigma_t \varepsilon_{s,t+1}.
\end{aligned} \tag{39}$$

Therefore, the equilibrium solutions for the log price/consumption ratio and risk-free rate are identical to the Bansal and Yaron (2004) model.

A.3.1 The dividend claim. We conjecture that the log price/dividend ratio is an affine function of the state variables, x_t, σ_t^2, and s_t:

$$z_{m,t} = A_{0,m} + A_{1,m} x_t + A_{2,m} \sigma_t^2 + A_{3,m} s_t. \tag{40}$$

The coefficients $A_{0,m}$, $A_{1,m}$, $A_{2,m}$, and $A_{3,m}$ are computed using the method of undetermined coefficients as described below.

The Euler equation for the observable return on the aggregate dividend claim, $r_{m,t+1}$, is

$$E_t \left[\exp\left(\theta \log \delta - \frac{\theta}{\psi} \Delta c_{t+1} + (\theta - 1) r_{c,t+1} + r_{m,t+1}\right)\right] = 1. \tag{41}$$

Substituting the expression for $r_{m,t+1}$ from Equation (9) into the above Euler condition, we have

$$E_t[\exp(\theta \log \delta - \frac{\theta}{\psi} \mu_c - \frac{\theta}{\psi} x_t - \frac{\theta}{\psi} \sigma_t \varepsilon_{c,t+1} + (\theta - 1)\kappa_0 + (\theta - 1)\kappa_1 A_0$$
$$+ (\theta - 1)\kappa_1 A_1 \rho_x x_t + (\theta - 1)\kappa_1 A_1 \psi_x \sigma_t \varepsilon_{x,t+1}$$

$$
\begin{aligned}
&+(\theta-1)\kappa_1 A_2(1-\upsilon)\sigma^2+(\theta-1)\kappa_1 A_2\upsilon\sigma_t^2+(\theta-1)\kappa_1 A_2\sigma_w\varepsilon_{\sigma,t+1}\\
&-(\theta-1)A_0-(\theta-1)A_1x_t-(\theta-1)A_2\sigma_t^2\\
&+(\theta-1)\mu_c+(\theta-1)x_t+(\theta-1)\sigma_t\varepsilon_{c,t+1}\\
&+\kappa_{0,m}+\kappa_{1,m}A_{0,m}+\kappa_{1,m}A_{1,m}\rho_x x_t+\kappa_{1,m}A_{1,m}\psi_x\sigma_t\varepsilon_{x,t+1}+\kappa_{1,m}A_{2,m}(1-\upsilon)\sigma^2\\
&+\kappa_{1,m}A_{2,m}\upsilon\sigma_t^2+\kappa_{1,m}A_{2,m}\sigma_w\varepsilon_{\sigma,t+1}+\kappa_{1,m}A_{3,m}\lambda_{sx}x_t+\kappa_{1,m}A_{3,m}\rho_s s_t\\
&+\kappa_{1,m}A_{3,m}\psi_s\sigma_t\varepsilon_{s,t+1}-A_{0,m}-A_{1,m}x_t-A_{2,m}\sigma_t^2-A_{3,m}s_t\\
&+\mu_c+(1+\lambda_{sx})x_t+(\rho_s-1)s_t+\sigma_t\varepsilon_{c,t+1}+\psi_s\sigma_t\varepsilon_{s,t+1})]\\
&\quad=1.
\end{aligned}
$$

Using the assumed conditional log-normality of the stochastic processes, the left-hand side of the above expression simplifies to

$$
\begin{aligned}
&\exp\Bigg(\theta\log\delta+\left(-\frac{\theta}{\psi}+\theta\right)\mu_c+(\theta-1)\kappa_0+(\theta-1)(\kappa_1-1)A_0+(\theta-1)\kappa_1A_2(1-\upsilon)\sigma^2\\
&+\kappa_{0,m}+(\kappa_{1,m}-1)A_{0,m}+\kappa_{1,m}A_{2,m}(1-\upsilon)\sigma^2\\
&+\left[\left(-\frac{\theta}{\psi}+\theta-1\right)+(\theta-1)(\kappa_1\rho_x-1)A_1+\left(\kappa_{1,m}\rho_x-1\right)A_{1,m}+(1+\lambda_{sx})\right]x_t\\
&+\left[\kappa_{1,m}A_{3,m}\lambda_{sx}\right]x_t+\left[\left(\kappa_{1,m}\rho_s-1\right)A_{3,m}+\rho_s-1\right]s_t\\
&+\left[(\theta-1)(\kappa_1\upsilon-1)A_2+\left(\kappa_{1,m}\upsilon-1\right)A_{2,m}\right]\sigma_t^2\\
&+0.5\left\{\left(-\frac{\theta}{\psi}+\theta\right)^2\sigma_t^2+\left[\kappa_{1,m}A_{3,m}+1\right]^2\psi_s^2\sigma_t^2\right.\\
&+\left.\left[(\theta-1)\kappa_1A_1+\kappa_{1,m}A_{1,m}\right]^2\psi_x^2\sigma_t^2+\left[(\theta-1)\kappa_1A_2+\kappa_{1,m}A_{2,m}\right]^2\sigma_w^2\right\}\Bigg)\\
&\quad=1.
\end{aligned}
\tag{42}
$$

Since the Euler equation (42) must hold for all values of the state variables, we have

$$
\left(\kappa_{1,m}\rho_s-1\right)A_{3,m}+\rho_s-1=0
$$

$$
A_{3,m}=\frac{\rho_s-1}{1-\kappa_{1,m}\rho_s}
\tag{43}
$$

$$
\left(-\frac{\theta}{\psi}+\theta-1\right)+(\theta-1)(\kappa_1\rho_x-1)A_1+\left(\kappa_{1,m}\rho_x-1\right)A_{1,m}+\kappa_{1,m}A_{3,m}\lambda_{sx}+1+\lambda_{sx}=0
$$

$$
A_{1,m}=\frac{1-\frac{1}{\psi}+\lambda_{sx}(1+\kappa_{1,m}A_{3,m})}{1-\kappa_{1,m}\rho_x}
$$

$$
\begin{aligned}
&(\theta-1)(\kappa_1\upsilon-1)A_2+\left(\kappa_{1,m}\upsilon-1\right)A_{2,m}+0.5\left\{\left(-\frac{\theta}{\psi}+\theta\right)^2\right.\\
&\left.+\left[\kappa_{1,m}A_{3,m}+1\right]^2\psi_s^2+\left[(\theta-1)\kappa_1A_1+\kappa_{1,m}A_{1,m}\right]^2\psi_x^2\right\}\\
&\quad=0
\end{aligned}
\tag{44}
$$

$$A_{2,m} = \frac{(\theta - 1)(\kappa_1 \upsilon - 1) A_2 + C}{1 - \kappa_{1,m} \upsilon}$$
$$C = 0.5\left\{ \left(-\frac{\theta}{\psi} + \theta\right)^2 + \left[\kappa_{1,m} A_{3,m} + 1\right]^2 \psi_s^2 + \left[(\theta - 1)\kappa_1 A_1 + \kappa_{1,m} A_{1,m}\right]^2 \psi_x^2 \right\}$$

$$\theta \log \delta + \left(-\frac{\theta}{\psi} + \theta\right) \mu_c + (\theta - 1)\kappa_0 + (\theta - 1)(\kappa_1 - 1) A_0 + (\theta - 1)\kappa_1 A_2 (1 - \upsilon) \sigma^2 + \kappa_{0,m} + (\kappa_{1,m} - 1) A_{0,m} + \kappa_{1,m} A_{2,m} (1 - \upsilon) \sigma^2 + 0.5 \left[(\theta - 1)\kappa_1 A_2 + \kappa_{1,m} A_{2,m}\right]^2 \sigma_w^2 = 0 \tag{45}$$

$$A_{0,m} = \frac{\theta \log \delta + \left(-\frac{\theta}{\psi} + \theta\right) \mu_c + (\theta - 1)\kappa_0 + (\theta - 1)\left(\kappa_1 - 1\right) A_0}{1 - \kappa_{1,m}} + \frac{(\theta - 1)\kappa_1 A_2 (1 - \upsilon) \sigma^2 + \kappa_{0,m} + \kappa_{1,m} A_{2,m} (1 - \upsilon) \sigma^2 + 0.5 \left[(\theta - 1)\kappa_1 A_2 + \kappa_{1,m} A_{2,m}\right]^2 \sigma_w^2}{1 - \kappa_{1,m}}. \tag{46}$$

A.3.2 Latent state variables in terms of observable variables.

We have

$$z_{m,t} = A_{0,m} + A_{1,m} x_t + A_{2,m} \sigma_t^2 + A_{3,m} s_t$$
$$r_{f,t} = A_{0,f} + A_{1,f} x_t + A_{2,f} \sigma_t^2.$$

The above equations may be inverted to express the unobservable state variables, x_t and σ_t^2, in terms of the observables, $z_{m,t}$, $r_{f,t}$, and s_t.

Define

$$D \equiv A_{1,m} A_{2,f} - A_{1,f} A_{2,m}.$$

We have

$$x_t = \alpha_0 + \alpha_1 r_{f,t} + \alpha_2 z_{m,t} + \alpha_3 s_t$$
$$\alpha_0 = \frac{A_{0,f} A_{2,m} - A_{0,m} A_{2,f}}{D}$$
$$\alpha_1 = \frac{-A_{2,m}}{D}$$
$$\alpha_2 = \frac{A_{2,f}}{D}$$
$$\alpha_3 = \frac{-A_{3,m} A_{2,f}}{D}$$

$$\sigma_t^2 = \beta_0 + \beta_1 r_{f,t} + \beta_2 z_{m,t} + \beta_3 s_t$$
$$\beta_0 = \frac{A_{0,m} A_{1,f} - A_{1,m} A_{0,f}}{D}$$
$$\beta_1 = \frac{A_{1,m}}{D}$$
$$\beta_2 = \frac{-A_{1,f}}{D}$$
$$\beta_3 = \frac{A_{1,f} A_{3,m}}{D}.$$

Now, from Equations (7), (8), and (10), the pricing kernel is given by the expression

$$m_{t+1} = \left(\theta \log \delta + (\theta - 1)\left[\kappa_0 + (\kappa_1 - 1)\, A_0\right]\right) + \left(-\frac{\theta}{\psi} + (\theta - 1)\right) \Delta c_{t+1} + (\theta - 1)\kappa_1 A_1 x_{t+1} + (\theta - 1)\kappa_1 A_2 \sigma_{t+1}^2 - (\theta - 1) A_1 x_t - (\theta - 1) A_2 \sigma_t^2.$$

Substituting the expressions for x_t and σ_t^2 into the above expression for the pricing kernel, we have

$$m_{t+1} = c_1 + c_2 \Delta c_{t+1} + c_3 \left(r_{f,t+1} - \frac{1}{\kappa_1} r_{f,t}\right) + c_4 \left(z_{m,t+1} - \frac{1}{\kappa_1} z_{m,t}\right) + c_5 \left(s_{t+1} - \frac{1}{\kappa_1} s_t\right),$$

where

$$c_1 = \theta \log \delta + (\theta - 1)[\kappa_0 + (\kappa_1 - 1)\,(A_0 + A_1 \alpha_0 + A_2 \beta_0)]$$
$$c_2 = -\frac{\theta}{\psi} + (\theta - 1) = -\gamma$$
$$c_3 = (\theta - 1)\kappa_1 [A_1 \alpha_1 + A_2 \beta_1]$$
$$c_4 = (\theta - 1)\kappa_1 [A_1 \alpha_2 + A_2 \beta_2]$$
$$c_5 = (\theta - 1)\kappa_1 [A_1 \alpha_3 + A_2 \beta_3].$$

A.3.3 Predictive implications for the equity premium and consumption and dividend growth.

Equation (3) implies

$$E_t\left(\Delta c_{t+1}\right) = \mu_c + x_t,$$

and Equation (4) implies

$$E_t\left(\Delta d_{t+1}\right) = \mu_c + (1 + \lambda_{sx})\, x_t + (\rho_s - 1)\, s_t.$$

Equations (9), (24), and (40) imply that the equilibrium market return is given by

$$r_{m,t+1} = \kappa_{0,m} + \kappa_{1,m}\left(A_{0,m} + A_{1,m} x_{t+1} + A_{2,m}\sigma_{t+1}^2 + A_{3,m} s_{t+1}\right) - \left(A_{0,m} + A_{1,m} x_t + A_{2,m}\sigma_t^2 + A_{3,m} s_t\right) + \mu_c + (1 + \lambda_{sx})\, x_t + (\rho_s - 1)\, s_t + \sigma_t \varepsilon_{c,t+1} + \psi_s \sigma_t \varepsilon_{s,t+1}.$$

Taking conditional expectation of the two sides of the above equation, and using Equations (1), (2), and (23), we have

$$E_t\left(r_{m,t+1}\right) = \kappa_{0,m} + \left(\kappa_{1,m} - 1\right) A_{0,m} + \mu_c + \kappa_{1,m} A_{2,m}\,(1 - \upsilon)\,\sigma^2 + \left[\left(\kappa_{1,m}\rho_x - 1\right) A_{1,m} + \kappa_{1,m} A_{3,m}\lambda_{sx} + 1 + \lambda_{sx}\right] x_t + \left(\kappa_{1,m}\upsilon - 1\right) A_{2,m}\sigma_t^2 + \left[\left(\kappa_{1,m}\rho_s - 1\right) A_{3,m} + \rho_s - 1\right] s_t.$$

Therefore, the expected equity premium is given by

$$E_t\left(r_{m,t+1} - r_{f,t}\right) = E_0 + E_1 \sigma_t^2,$$

where $E_1 = \left(\kappa_{1,m}\upsilon - 1\right) A_{2,m} + (\theta - 1)(\kappa_1 \upsilon - 1) A_2 + 0.5\left(\left(-\frac{\theta}{\psi} + \theta - 1\right)^2 + (\theta - 1)^2 \kappa_1^2 A_1^2 \psi_x^2\right).$

A.4 Estimation of time-series parameters of the co-integrated model

In this specification, there are nine parameters to be estimated—μ_c, ρ_x, ψ_x, σ, υ, σ_w, λ_{sx}, ρ_s, and ψ_s.

We have

$$E\left(\Delta c_{t+1}\right) = \mu_c. \tag{47}$$

Also,

$$\begin{aligned} Var\left(\Delta c_{t+1}\right) &= Var\left(x_t\right) + Var\left(\sigma_t \varepsilon_{c,t+1}\right) + 2Cov(x_t, \sigma_t \varepsilon_{c,t+1}) \\ &= Var\left(x_t\right) + \sigma^2 + 0 \\ &= \frac{\psi_x^2 \sigma^2}{1-\rho_x^2} + \sigma^2 \end{aligned} \tag{48}$$

and

$$Cov(\Delta c_{t+1}, \Delta c_{t+2}) = \rho_x \frac{\psi_x^2 \sigma^2}{1-\rho_x^2} \tag{49}$$

$$Cov(\Delta c_{t+1}, \Delta c_{t+3}) = \rho_x^2 \frac{\psi_x^2 \sigma^2}{1-\rho_x^2}. \tag{50}$$

From the specification of the dividend growth process, we have

$$\begin{aligned} Var\left(\Delta d_{t+1}\right) &= (1+\lambda_{sx})^2\, Var\left(x_t\right) + (\rho_s - 1)^2\, Var\left(s_t\right) \\ &\quad + (1+\psi_s^2)\sigma^2 + 2\,(1+\lambda_{sx})\,(\rho_s - 1)\, Cov(x_t, s_t), \end{aligned} \tag{51}$$

where $Var\left(x_t\right) = \frac{\psi_x^2 \sigma^2}{1-\rho_x^2}$, $Cov(x_t, s_t) = \frac{\lambda_{sx}\rho_x}{1-\rho_x \rho_s} Var\left(x_t\right)$,

$$\text{and } \; Var\left(s_t\right) = \frac{\lambda_{sx}^2 Var\left(x_t\right) + \psi_s^2 \sigma^2 + \frac{2\lambda_{sx}^2 \rho_x \rho_s Var(x_t)}{1-\rho_x \rho_s}}{1-\rho_s^2}.$$

Also,

$$\begin{aligned} Cov(\Delta d_{t+1}, \Delta d_{t+2}) &= (1+\lambda_{sx})^2\, Cov(x_{t+1}, x_t) + (\rho_s - 1)^2\, Cov(s_{t+1}, s_t) \\ &\quad + (1+\lambda_{sx})\,(\rho_s - 1)\left[Cov(x_{t+1}, s_t) + Cov(x_t, s_{t+1})\right] \\ &\quad + (\rho_s - 1)\, \psi_s Cov(s_{t+1}, \sigma_t \varepsilon_{s,t+1}), \end{aligned} \tag{52}$$

where $Cov(x_{t+1}, x_t) = \rho_x Var\left(x_t\right)$, $Cov(s_{t+1}, s_t) = \lambda_{sx} Cov(x_t, s_t) + \rho_s Var\left(s_t\right)$, $Cov(x_t, s_{t+1}) = \lambda_{sx} Var\left(x_t\right) + \rho_s Cov(x_t, s_t)$, $Cov(x_{t+1}, s_t) = \rho_x Cov(x_t, s_t)$, and $Cov(s_{t+1}, \sigma_t \varepsilon_{s,t+1}) = \psi_s \sigma^2$.

Finally,

$$Cov(\Delta c_{t+1}, \Delta d_{t+1}) = (1+\lambda_{sx})\, Var\left(x_t\right) + (\rho_s - 1)\, Cov(x_t, s_t) + \sigma^2. \tag{53}$$

Equations (47)–(53) give seven moment restrictions.

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